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Exponentially closed fields and the conjecture on intersections with tori

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ABSTRACT

We give an axiomatization of the class **ECF** of exponentially closed fields, which includes the pseudo-exponential fields previously introduced by the second author, and show that it is superstable over its interpretation of arithmetic. Furthermore, **ECF** is exactly the elementary class of the pseudo-exponential fields if and only if the Diophantine conjecture CIT on atypical intersections of tori with subvarieties is true.

1. Introduction

1.1. Pseudo-exponential fields

In [20], the second author introduced a class of exponential fields he called *pseudo-exponential fields*, as the class of models $\langle F; +, \cdot, \exp\rangle$ of the following five axioms, including the statement of the well-known Schanuel conjecture.

1. **ELA-field:** F is an algebraically closed field of characteristic zero, and its exponential map exp is a homomorphism from its additive group to its multiplicative group, which is surjective.

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2. **Standard kernel:** The kernel of the exponential map is an infinite cyclic group generated by a transcendental element τ.
3. **Schanuel property:** The *predimension function*

$$\delta(\bar{x}) := \operatorname{td}\big(\bar{x}, \exp(\bar{x})\big) - \operatorname{ldim}_{\mathbb{Q}}(\bar{x})$$

satisfies $\delta(\bar{x}) \geqslant 0$ for all tuples $\bar{x}$ from F.
4. **Strong exponential–algebraic closedness:** If V is a rotund, additively and multiplicatively free subvariety of $\mathbb{G}_a^n \times \mathbb{G}_m^n$ defined over F and of dimension n, and $\bar{a}$ is a finite tuple from F, then there is $\bar{x}$ in F such that $(\bar{x}, e^{\bar{x}}) \in V$ and is generic in V over $\bar{a}$.
5. **Countable closure property:** For each finite subset X of F, the exponential algebraic closure $\operatorname{ecl}^F(X)$ of X in F is countable.

We call any model of axiom 1 an *ELA-field*: E for exponentiation, L for the surjectivity (every non-zero element has a logarithm) and A for algebraically closed. Precise definitions of the terms in axioms 4 and 5 are given later, in Sections 3.7 and 3.6 respectively. Intuitively, axiom 4 says that any system of equations which can have a solution in some suitable exponential extension field does already have a solution in F. It is the analogue for exponential fields of the algebraic closedness axiom for fields, which says that any polynomial equation (or, equivalently, any system of polynomial equations) which can have a solution in some extension field already has a solution in F. Axiom 5 says that such a system should only have countably many solutions.

We denote by $\mathbf{ECF_{SK}}$ the class of models of axioms 1–4, and call the models *Exponentially-Closed Fields with Standard Kernel*. We also denote by $\mathbf{ECF_{SK,CCP}}$ the class of models of axioms 1–5. (In [20] the same classes were called $\mathcal{EC}^*_{st}$ and $\mathcal{EC}^*_{st,ccp}$.)

The main theorem of [20] was that $\mathbf{ECF_{SK,CCP}}$ has exactly one model in each uncountable cardinality, up to isomorphism. This categoricity theorem was not proved entirely in one paper. The proof depends on the main result from [21], and corrections to the two papers appeared in [2] and [1]. The proof also uses the model-theoretic technique of *quasiminimal excellent classes* which were developed for this purpose in [19], and further developed and simplified in [10] and [3].

We write $\mathbb{B}$ for the model of $\mathbf{ECF_{SK}}$ of cardinality $2^{\aleph_0}$. The complex exponential field $\mathbb{C}_{\exp}$ is known to satisfy axioms 1 and 2 (trivially) and 5 (less trivially). Schanuel's conjecture is a fundamental conjecture of transcendental number theory. Since strong exponential–algebraic closedness is very natural from the model-theoretic point of view, it makes sense to conjecture that it holds for $\mathbb{C}_{\exp}$. Together, these two conjectures are therefore equivalent to the assertion that $\mathbb{C}_{\exp}$ is isomorphic to $\mathbb{B}$.

The axioms for $\mathbf{ECF_{SK}}$ are not all first-order expressible, but they can all be expressed in the logic $L_{\omega_1,\omega}$, which allows countable conjunctions of formulas. In fact, $\mathbf{ECF_{SK}}$ can also be viewed as the class of models of a complete first-order theory which omit the type of a non-standard integer [11]. We denote the first-order theory by $T_{\mathbb{B}}$. (The countable closure property is not expressible in $L_{\omega_1,\omega}$, but can be expressed using the quantifier Qx: *there exist uncountably many x such that ...*. However, we will make no use of that axiom for the rest of the paper.)

While $\mathbf{ECF_{SK}}$ is a well-behaved class of structures, it is not an elementary class. Understanding the first-order theory $T_{\mathbb{B}}$ should give much more information which we believe may be useful in understanding the analytic geometry of $\mathbb{C}_{\exp}$, perhaps even without assuming that $\mathbb{C}_{\exp}$ is isomorphic to $\mathbb{B}$. This paper seeks to give an understanding of $T_{\mathbb{B}}$. Using the categoricity theorem for $\mathbf{ECF_{SK,CCP}}$ we see that $T_{\mathbb{B}}$ is the complete first-order theory of $\mathbb{B}$, hence our notation. However, the results of this paper do not depend on the categoricity theorem.

An obvious obstacle to the goal of understanding $T_{\mathbb{B}}$ comes from the integers. For any exponential field F, we will write $\ker(F) = \{x \in F \mid \exp(x) = 1\}$, the kernel of the exponential map. We also define