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Asymmetric regular types

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ABSTRACT

We study asymmetric regular global types $\mathfrak{p} \in S_1(\mathfrak{C})$. If $\mathfrak{p}$ is regular and A-asymmetric then there exists a strict order such that Morley sequences in $\mathfrak{p}$ over A are strictly increasing (we allow Morley sequences to be indexed by elements of a linear order). We prove that for any small model $M \supseteq A$ maximal Morley sequences in $\mathfrak{p}$ over A consisting of elements of M have the same (linear) order type, denoted by $\mathrm{Inv}_{\mathfrak{p},A}(M)$. In the countable case we determine all possibilities for $\mathrm{Inv}_{\mathfrak{p},A}(M)$: either it can be any countable linear order, or in any $M \supseteq A$ it is a dense linear order (provided that it has at least two elements). Then we study relationship between $\mathrm{Inv}_{\mathfrak{p},A}(M)$ and $\mathrm{Inv}_{\mathfrak{q},A}(M)$ when $\mathfrak{p}$ and $\mathfrak{q}$ are strongly regular, A-asymmetric, and such that $\mathfrak{p}_{\restriction A}$ and $\mathfrak{q}_{\restriction A}$ are not weakly orthogonal. We distinguish two kinds of non-orthogonality: bounded and unbounded. In the bounded case we prove that $\mathrm{Inv}_{\mathfrak{p},A}(M)$ and $\mathrm{Inv}_{\mathfrak{q},A}(M)$ are either isomorphic or anti-isomorphic. In the unbounded case, $\mathrm{Inv}_{\mathfrak{p},A}(M)$ and $\mathrm{Inv}_{\mathfrak{q},A}(M)$ may have distinct cardinalities but we prove that their Dedekind completions are either isomorphic or anti-isomorphic. We provide examples of all four situations.

The concept of regularity for global types in an arbitrary first order theory was introduced in [4]. It was motivated by certain properties of regular (stationary) types in stable theories. The motivation behind the definition is the following. We operate in the monster model $\mathfrak{C}$ of a complete first-order theory T. By $A, B, \dots$ we denote small subsets and by $M, M_0, \dots$ small elementary submodels of the monster. Let $\mathfrak{p}$ be a global A-invariant type. $\mathfrak{p}$ induces a division of all definable subsets: those defined by a formula belonging to $\mathfrak{p}$ are considered to be 'large' subsets, the others are 'small'. The division induces a naturally defined operation on the power set of $\mathfrak{C}$: $\mathrm{cl}_{\mathfrak{p}}^A(X)$ is the union of all small AX-definable subsets. The regularity of $\mathfrak{p}$ over A means that $\mathrm{cl}_{\mathfrak{p}}^B$ is a closure operation on the power set of the locus of $\mathfrak{p}_{\restriction B}$ for all $B \supseteq A$ (this is explained in detail in Section 2). It turns out that there are two essentially distinct kinds of regular types: symmetric

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and asymmetric. For types of symmetric kind $\mathrm{cl}_{\mathfrak{p}}^{B}$ is a pregeometry operation carrying a naturally defined notion of dimension. In the generically stable case they share several nice properties of regular types in a stable theory; this was investigated in more detail in [5]. In the asymmetric case some $\mathrm{cl}_{\mathfrak{p}}^{B}$ is a proper closure operation (the exchange fails). The main consequence established in [4] is the existence of a definable partial order which orders Morley sequences increasingly. In this paper we study asymmetric regular types in more detail. It turns out that in most results we do not use the full strength of the original regularity assumption. What suffices is that the type is weakly regular over A ($\mathrm{cl}_{\mathfrak{p}}^{A}$ is a proper closure operation on the locus of $\mathfrak{p}_{\restriction A}$) and A-asymmetric; see Definitions 2.1 and 2.3. The main consequence of these assumptions from [4] is the existence of a definable partial order which orders Morley sequences increasingly. That fact is re-proved in the first part of the following theorem.

Theorem 1. *Suppose that $\mathfrak{p}$ is weakly regular and A-asymmetric. There is an A-definable partial order such that any Morley sequence in $\mathfrak{p}$ over A is strictly increasing. For any $X \subseteq \mathfrak{C}$ the order type of any maximal Morley sequence (in $\mathfrak{p}$ over A) consisting of elements of X does not depend on the particular choice of the sequence.*

Thus instead of having a dimension of a type in a model, we have another invariant: the linear order type of a maximal Morley sequence, denoted by $\mathrm{Inv}_{\mathfrak{p},A}(M)$. The partial order $\leq$ mentioned in the theorem is not uniquely determined, and all of them, when considered on $\mathfrak{p}_{\restriction A}(\mathfrak{C})$, have a strong combinatorial property. Essentially '$\{x, y\}$ cannot be ordered into a Morley sequence in $\mathfrak{p}$ over A' is an equivalence relation on the locus, and its classes are linearly ordered by $\leq$ in the same way for any choice of $\leq$. The equivalence relation is relatively $\bigvee$-definable over A on the locus of $\mathfrak{p}_{\restriction A}$; the class of $a \models \mathfrak{p}_{\restriction A}$ is denoted by $\mathcal{E}_{\mathfrak{p}}(a)$. For any M the naturally induced order on all the classes meeting M is also linear and its order type is $\mathrm{Inv}_{\mathfrak{p},A}(M)$. We investigate possibilities for $\mathrm{Inv}_{\mathfrak{p},A}(M)$. Two properties of the regular type are relevant for. The first is simplicity, i.e. the relative definability of '(x, y) is a Morley sequence in $\mathfrak{p}$ over A' which is essentially equivalent to $\mathrm{Inv}_{\mathfrak{p},A}(M)$ being represented by a type-definable object in M^{eq}. The second property is convexity over A: whether the order witnessing the A-asymmetric regularity can be chosen such that the locus of $\mathfrak{p}_{\restriction A}$ is a convex subset of $\mathfrak{C}$. We prove:

Theorem 2. *Suppose that $\mathfrak{p}$ is weakly regular and A-asymmetric.*

(i) If $\mathfrak{p}$ is simple and convex over A and $\mathrm{Inv}_{\mathfrak{p},A}(M)$ has at least two elements then $\mathrm{Inv}_{\mathfrak{p},A}(M)$ is a dense linear order (with or without endpoints).
(ii) If both T and A are countable and $\mathfrak{p}$ is simple and non-convex over A then $\mathrm{Inv}_{\mathfrak{p},A}(M)$ can be any countable linear order.
(iii) If both T and A are countable and $\mathfrak{p}$ is not simple over A then $\mathrm{Inv}_{\mathfrak{p},A}(M)$ can be any countable linear order.

Next we study general properties of orthogonality of regular types. We prove:

Theorem 3. *A regular asymmetric type is orthogonal to any invariant symmetric type. $\not\perp$ is an equivalence relation on the set of asymmetric, regular types.*

Now, assume that $\mathfrak{p}, \mathfrak{q}$ are regular, A-asymmetric types and that the corresponding restrictions are $\not\perp^{w}$. It is natural to determine the relationship between $\mathrm{Inv}_{\mathfrak{p},A}(M)$ and $\mathrm{Inv}_{\mathfrak{q},A}(M)$. In general, there may be no connection between them: in Example 6.1 one of them is empty and the other any dense linear order chosen in advance. But assuming in addition that the types are strongly regular, or at least convex in some cases, we get a strong connection. There are two kinds of $\not\perp^{w}$ which we call bounded and unbounded. In the bounded case there is an A-invariant bijection between $\mathcal{E}_{\mathfrak{p}}$- and $\mathcal{E}_{\mathfrak{q}}$-classes, in which case $\mathrm{Inv}_{\mathfrak{p},A}(M)$