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Categorical characterizations of the natural numbers require primitive recursion

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ABSTRACT

Simpson and Yokoyama (2013) [9] asked whether there exists a characterization of the natural numbers by a second-order sentence which is provably categorical in the theory RCA_0^*. We answer in the negative, showing that for any characterization of the natural numbers which is provably true in WKL_0^*, the categoricity theorem implies Σ_1^0 induction.

On the other hand, we show that RCA_0^* does make it possible to characterize the natural numbers categorically by means of a set of second-order sentences. We also show that a certain Π_2^1-conservative extension of RCA_0^* admits a provably categorical single-sentence characterization of the naturals, but each such characterization has to be inconsistent with $\mathsf{WKL}_0^* + \text{superexp}$.

Inspired by a question of Väänänen (see e.g. [11] for some related work), Simpson and the second author [9] studied various second-order characterizations of $\langle \mathbb{N}, S, 0\rangle$, with the aim of determining the reverse-mathematical strength of their respective categoricity theorems. One of the general conclusions is that the strength of a categoricity theorem depends heavily on the characterization. Strikingly, however, each of the categoricity theorems considered in [9] implies RCA_0, even over the much weaker base theory RCA_0^*, that is, RCA_0 with Σ_1^0 induction replaced by Δ_0^0 induction in the language with exponentiation. (For RCA_0^*, see [8].)

This leads to the following question.

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Question 1 *([9, Question 5.3, slightly rephrased]).* Does RCA_0^* prove the existence of a second-order sentence or set of sentences T such that $\langle \mathbb{N}, S, 0\rangle$ is a model of T and all models of T are isomorphic to $\langle \mathbb{N}, S, 0\rangle$? One may also consider the same question with RCA_0^* replaced by Π_2^0-conservative extensions of RCA_0^*.

Naturally, to have any hope of characterizing infinite structures categorically, second-order logic has to be interpreted according to the *standard* semantics (sometimes also known as strong or Tarskian semantics), as opposed to the *general* (or Henkin) semantics. In other words, a second-order quantifier $\forall X$ really means "for *all* subsets of the universe" (or, as we would say in a set-theoretic context, "for all elements of the power set of the universe").

Question 1 admits multiple versions depending on whether we focus on RCA_0^* or consider other Π_2^0-equivalent theories and whether we want the characterizations of the natural numbers to be sentences or sets of sentences. The most basic version, restricted to RCA_0^* and single-sentence characterizations, would read as follows:

Question 2. Does there exist a second-order sentence ψ in the language with one unary function f and one constant c such that RCA_0^* proves: (i) $\langle \mathbb{N}, S, 0\rangle \models \psi$, and (ii) for every $\langle A, f, c\rangle$, if $\langle A, f, c\rangle \models \psi$, then there exists an isomorphism between $\langle \mathbb{N}, S, 0\rangle$ and $\langle A, f, c\rangle$?

We answer Question 2 in the negative. In fact, characterizing $\langle \mathbb{N}, S, 0\rangle$ not only up to isomorphism, but even just up to *equicardinality of the universe*, requires the full strength of RCA_0. More precisely:

Theorem 1. *Let ψ be a second-order sentence in the language with one unary function f and one individual constant c. If WKL_0^* proves that $\langle \mathbb{N}, S, 0\rangle \models \psi$, then over RCA_0^* the statement "for every $\langle A, f, c\rangle$, if $\langle A, f, c\rangle \models \psi$, then there exists a bijection between $\mathbb{N}$ and A" implies RCA_0.*

Since RCA_0 is equivalent over RCA_0^* to a statement expressing the correctness of defining functions by primitive recursion [8, Lemma 2.5], Theorem 1 may be intuitively understood as saying that, for provably true single-sentence characterizations at least, "categorical characterizations of the natural numbers require primitive recursion".

Do less stringent versions of Question 1 give rise to "exceptions" to this general conclusion? As it turns out, they do. Firstly, characterizing the natural numbers by a *set* of sentences is already possible in RCA_0^*, in the following sense (for a precise statement of the theorem, see Section 4):

Theorem 2. *There exists a Δ_0-definable (and polynomial-time recognizable) set Ξ of $\Sigma_1^1 \wedge \Pi_1^1$ sentences such that RCA_0^* proves: for every $\langle A, f, c\rangle$, $\langle A, f, c\rangle$ satisfies all $\xi \in \Xi$ if and only if $\langle A, f, c\rangle$ is isomorphic to $\langle \mathbb{N}, S, 0\rangle$.*

Secondly, even a single-sentence characterization is possible in a Π_2^1-conservative extension of RCA_0^*, at least if one is willing to consider rather peculiar theories:

Theorem 3. *There is a Σ_2^1 sentence which is a categorical characterization of $\langle \mathbb{N}, S, 0\rangle$ provably in the theory $\mathsf{RCA}_0^* + \neg\mathrm{WKL}$.*

Theorem 3 is not quite satisfactory, as the theory and characterization it speaks of are false in $\langle \omega, \mathcal{P}(\omega)\rangle$. So, another natural question to ask is whether a single-sentence characterization of the natural numbers can be provably categorical in a *true* Π_2^0-conservative extension of RCA_0^*. We show that under an assumption just a little stronger than Π_2^0-conservativity, the characterization from Theorem 3 is actually "as true as possible":