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Notes on some second-order systems of iterated inductive definitions and Π^1_1-comprehensions and relevant subsystems of set theory

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ABSTRACT

Pohlers's ordinal analysis in his monograph [12] contains some flaws and thereby ends up with incorrect proof-theoretic ordinals of several systems. The present paper determines their correct proof-theoretic ordinals and also supplements [12] with the ordinal analysis of some other relevant impredicative systems.

1. Introduction

Iterated inductive definitions and Π^1_1-comprehensions have been playing an important role in impredicative proof theory, and a number of significant developments on this subject were obtained in the late 20th century; cf. [5] and [14]. These results and developments are summed up and even extended in Pohlers's monograph [12], which concludes with a fairly comprehensive list of the proof-theoretic ordinals of various systems of iterated inductive definitions and Π^1_1-comprehensions as well as relevant subsystems of set theory. However, Pohlers's analysis there contains some flaws and thereby ends up with incorrect proof-theoretic ordinals of the following systems:

$$\left(\mathsf{ID}^2_\nu\right)_0, \quad \left(\Pi^1_1\text{-}\mathsf{CA}_\nu\right)_0, \quad \mathsf{ID}^2_\nu, \quad \Pi^1_1\text{-}\mathsf{CA}_\nu, \quad \mathsf{BID}^2_\nu, \quad \mathsf{ID}^2_\nu + \mathrm{Bi}, \quad \Pi^1_1\text{-}\mathsf{CA}_\nu + \mathrm{Bi}, \quad \mathsf{KPI}^r_\nu. \tag{1}$$

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In addition, his proof for the upper bounds of the proof-theoretic ordinals of $\mathsf{W\text{-}KPI}_\nu$ and $\mathsf{W\text{-}KPI}$ is also flawed, although his estimate of their proof-theoretic ordinals is in fact correct. The expositions and details of these flaws will be given in Section 4.[1]

The objective of the present paper is to determine the correct proof-theoretic ordinals of those systems listed in (1) above, to amend Pohlers's ordinal analysis of $\mathsf{W\text{-}KPI}_\nu$ and $\mathsf{W\text{-}KPI}$, and to supplement [12] with well-ordering proofs for these systems, which are not given in [12], and with the ordinal analyses of some other relevant impredicative systems.[2] The results of the present paper are summarized in the table in Section 11 of the present paper.

2. Ordinals

We first of all introduce an ordinal system in terms of which the proof-theoretic ordinals of the systems in question are expressed and on which Pohlers himself bases his ordinal analysis in [12]. This ordinal system was originally presented by Buchholz [4], and its detailed exposition can be found in [4, § 4] and [12, Ch. 3.4.4]. The present section lists the necessary and relevant facts about the ordinal system.

2.1. Preliminaries

We denote the class of ordinals by On. An ordinal $\alpha > 0$ is said to be *principal* when $\beta + \gamma < \alpha$ for all $\beta, \gamma < \alpha$. Let $\mathbb{H}$ denote the class of principal ordinals. Cantor's normal form theorem tells us that every ordinal is uniquely decomposed into the sum of finitely many principal ordinals $\alpha_0 \geq \cdots \geq \alpha_k$: this unique decomposition will be expressed as $\alpha =_{\mathrm{CNF}} \alpha_0 + \cdots + \alpha_k$.

Let $\lambda\alpha\lambda\beta.\varphi_\alpha\beta$ be the binary Veblen function; note that $\varphi_\alpha\beta$ is principal for all α and β. For each principal α, there uniquely exist β and γ such that $\alpha = \varphi_\beta\gamma$ and $\gamma < \alpha$. Hence, for each ordinal α, there uniquely exist $\beta_0, \dots, \beta_k$ and $\gamma_0, \dots, \gamma_k$ such that $\alpha =_{\mathrm{CNF}} \varphi_{\beta_0}\gamma_0 + \cdots + \varphi_{\beta_k}\gamma_k$ and $\gamma_i < \varphi_{\beta_i}\gamma_i$ for all $i \leq k$; this unique decomposition will be denoted by $\alpha =_{\mathrm{VNF}} \varphi_{\beta_0}\gamma_0 + \cdots + \varphi_{\beta_k}\gamma_k$.

An ordinal $\alpha > 0$ is called *strongly critical* when $\varphi_\beta\gamma < \alpha$ for all $\beta, \gamma < \alpha$. We denote the class of strongly critical ordinals by SC. It is know that $\alpha \in SC$ iff $\alpha =_{\mathrm{VNF}} \varphi_\alpha 0$; hence, $\alpha \notin SC$ iff $\alpha < \varphi_\alpha 0$.

We define the binary *fixed-point free* Veblen function $\overline{\varphi}$ (cf. [13, p. 42]) as follows:

$$\overline{\varphi}_\alpha\beta := \begin{cases} \varphi_\alpha(\beta+1) & \text{if } \beta = \gamma + n \text{ for some } n < \omega \text{ and } \gamma \text{ such that } \varphi_\alpha\gamma = \alpha \text{ or } \varphi_\alpha\gamma = \gamma \\ \varphi_\alpha\beta & \text{otherwise.} \end{cases}$$

We have $\alpha, \beta < \overline{\varphi}_\alpha\beta$ and $\overline{\varphi}_\alpha\beta \notin SC$ for all $\alpha, \beta \in On$, and $\lambda\beta.\overline{\varphi}_\alpha\beta$ is strictly increasing. It is easily shown that $\alpha \in SC$ iff $\overline{\varphi}_\beta\gamma < \alpha$ for all $\beta, \gamma < \alpha$ and $\alpha > 0$. Then, we can show the following:

$$\overline{\varphi}_\alpha\beta = \overline{\varphi}_\gamma\delta \quad \Leftrightarrow \quad \alpha = \gamma \text{ and } \beta = \delta$$

$$\overline{\varphi}_\alpha\beta < \overline{\varphi}_\gamma\delta \quad \Leftrightarrow \quad \begin{cases} \alpha = \gamma \text{ and } \beta < \delta, \text{ or} \\ \alpha < \gamma \text{ and } \beta < \overline{\varphi}_\gamma\delta, \text{ or} \\ \gamma < \alpha \text{ and } \overline{\varphi}_\alpha\beta \leq \delta. \end{cases}$$

[1] Systems of autonomously iterated inductive definitions such as $\mathsf{Aut\text{-}ID}$ and their relevant systems such as $\mathsf{Aut}\text{-}\Pi^1_1$ and $\mathsf{Aut\text{-}KPl}$ are also treated in [12]. The ordinal analyses of these systems were originally given by Rathjen [14], which is partly translated into English in [16]. Ordinal analyses of some other systems of the strength between Π^1_1-CA and Δ^1_2-CA + Bi (equivalent to KPi) than those treated in [12] are also found in [14].

[2] The proof-theoretic ordinals of ID^2_ν, Π^1_1-CA_ν and BID^2_ν were already determined by Jäger and Strahm [10] for a special case where $\nu = 1$ in terms of a different collapsing function; the present paper also provides direct proofs for their upper bounds, whereas Jäger and Strahm gave their upper bounds via the proof-theoretic analysis of other systems $\widetilde{\mathsf{E}\Omega}$ and $\mathsf{E}\Omega$.