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Set-theoretic geology ☆

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ABSTRACT

A *ground* of the universe V is a transitive proper class $W \subseteq V$, such that $W \models \mathrm{ZFC}$ and V is obtained by set forcing over W, so that $V = W[G]$ for some W-generic filter $G \subseteq \mathbb{P} \in W$. The model V satisfies the ground axiom GA if there are no such W properly contained in V. The model W is a *bedrock* of V if W is a ground of V and satisfies the ground axiom. The *mantle* of V is the intersection of all grounds of V. The *generic mantle* of V is the intersection of all grounds of all set-forcing extensions of V. The generic HOD, written gHOD, is the intersection of all HODs of all set-forcing extensions. The generic HOD is always a model of ZFC, and the generic mantle is always a model of ZF. Every model of ZFC is the mantle and generic mantle of another model of ZFC. We prove this theorem while also controlling the HOD of the final model, as well as the generic HOD. Iteratively taking the mantle penetrates down through the *inner* mantles to what we call the *outer core*, what remains when all outer layers of forcing have been stripped away. Many fundamental questions remain open.

The technique of forcing in set theory is customarily thought of as a method for constructing *outer* as opposed to *inner* models of set theory. A set theorist typically has a model of set theory V and constructs a larger model $V[G]$, the forcing extension, by adjoining a V-generic filter G over some partial order $\mathbb{P} \in V$.

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A switch in perspective, however, allows us to view forcing as a method of describing inner models as well. The idea is simply to search inwardly for how the model V might itself have arisen by forcing. Given a set-theoretic universe V, we consider the classes W over which V can be realized as a forcing extension $V = W[G]$ by some W-generic filter $G \subseteq \mathbb{P} \in W$. This change in viewpoint is the basis for a collection of questions leading to the topic we refer to as set-theoretic geology. In this article, we present some of the most interesting initial results in the topic, along with an abundance of open questions, many of which concern fundamental issues.

1. The mantle

We assume that the reader is familiar with the technique of forcing in set theory. Working in ZFC set theory and sometimes in GBC set theory, we suppose V is the universe of all sets. A class W is a *ground* of V, if W is a transitive class model of ZFC and V is obtained by set forcing over W, that is, if there is some forcing notion $\mathbb{P} \in W$ and a W-generic filter $G \subseteq \mathbb{P}$ such that $V = W[G]$. Laver [23] and independently Woodin [35,34] proved in this case that W is a definable class in V, using parameters in W, a result that was generalized by Hamkins to include many natural instances of class forcing (see Theorems 5 and 6). Building on these ideas, Hamkins and Reitz [11,29,30] introduced the following axiom.

Definition 1. The *ground axiom* GA is the assertion that the universe V is not obtained by set forcing over any strictly smaller ground model.

Because of the quantification over classes, the ground axiom assertion appears at first to be fundamentally second order in nature, but Reitz [30,29] proved that it is actually first-order expressible (an equivalent claim is implicit, independently, in [34]).

Definition 2. A class W is a *bedrock* for V if it is a ground of V and minimal with respect to the forcing-extension relation.

Since a ground of a ground is a ground, we may equivalently define that W is a bedrock of V if it is a ground of V and satisfies the ground axiom. Also, since by Fact 11 any inner model U of ZFC with $W \subseteq U \subseteq V$ for some ground W of V is both a forcing extension of W and a ground of V, we may equivalently define that W is a bedrock for V if it is a ground of V that is minimal with respect to inclusion among all grounds of V. It remains an open question whether there can be a model V having more than one bedrock model.

In this article, we attempt to carry the investigation deeper underground, bringing to light the structure of the grounds of the set-theoretic universe V and how they relate to the grounds of the forcing extensions of V. Continuing the geological metaphor, the principal new concept is:

Definition 3. The *mantle* M of a model of set theory is the intersection of all of its grounds.

The ground axiom can be reformulated as the assertion $V = \mathrm{M}$, that is, as the assertion that V is its own mantle. The mantle was briefly mentioned, unnamed, at the conclusion of [30], where the question was raised whether it necessarily models ZFC. Our main theorem in this article is a converse of sorts:

Main Theorem 4. *Every model of* ZFC *is the mantle of another model of* ZFC.

This theorem is a consequence of the more specific claims of Theorems 66 and 67, in which we are able not only to control the mantle of the target model, but also what we call the generic mantle, as well as the HOD and generic HOD. We begin by proving that the mantle, although initially defined with