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Borel structurability on the 2-shift of a countable group

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ABSTRACT

We show that for any infinite countable group G and for any free Borel action $G \curvearrowright X$ there exists an equivariant class-bijective Borel map from X to the free part $\mathrm{Free}(2^G)$ of the 2-shift $G \curvearrowright 2^G$. This implies that any Borel structurability which holds for the equivalence relation generated by $G \curvearrowright \mathrm{Free}(2^G)$ must hold *a fortiori* for all equivalence relations coming from free Borel actions of G. A related consequence is that the Borel chromatic number of $\mathrm{Free}(2^G)$ is the maximum among Borel chromatic numbers of free actions of G. This answers a question of Marks. Our construction is flexible and, using an appropriate notion of genericity, we are able to show that in fact the generic G-equivariant map to 2^G lands in the free part. As a corollary we obtain that for every $\epsilon > 0$, every free p.m.p. action of G has a free factor which admits a 2-piece generating partition with Shannon entropy less than ϵ. This generalizes a result of Danilenko and Park.

1. Introduction

Let G be a countably infinite discrete group. For a Polish space K, we equip $K^G = \prod_{g\in G} K$ with the product topology and we let G act on K^G via the left shift action: $(g \cdot w)(h) = w(g^{-1}h)$ for $g, h \in G$ and $w \in K^G$. We call K^G the **K-shift**. For $W \subseteq K^G$ we write $\overline{W}$ for the closure of W. The **free part of** K^G, denoted $\mathrm{Free}(K^G)$, is the set of points having trivial stabilizer:

$$\mathrm{Free}(K^G) = \{w \in K^G \,:\, \forall g \in G \; g \neq 1_G \Longrightarrow g \cdot w \neq w\}.$$

We mention that, unless $|K| = 1$, the set $\mathrm{Free}(K^G)$ is not closed in K^G. We will work almost exclusively with the 2-shift 2^G, where we use the convention that $2 = \{0, 1\}$.

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Let $G \curvearrowright X$ be a Borel action of G on a standard Borel space X. Our starting point is the well-known bijective correspondence

$$\{\text{Borel subsets of } X\} \longleftrightarrow \{G\text{-equivariant Borel maps from } X \text{ into } 2^G\},$$

which sends a Borel subset $A \subseteq X$ to the map $f_A : X \to 2^G$ given by $f_A(x)(g) = 1_{g \cdot A}(x)$, and whose inverse sends a G-equivariant Borel map $f : X \to 2^G$ to the set $A_f = \{x \in X : f(x)(1_G) = 1\}$. Since the map f_A encodes information not only about the set A, but also about each of its infinitely many translates $\{g \cdot A\}_{g \in G}$, it is not surprising that properties of f_A can depend very subtly on A. In this article, we provide a flexible construction, based on a construction of Gao, Jackson, and Seward [2], of subsets $A \subseteq X$ that yield G-equivariant Borel maps into the free part $\mathrm{Free}(2^G)$ of 2^G, under the assumption that the action $G \curvearrowright X$ is free. It is easy to see that freeness of $G \curvearrowright X$ is a necessary condition for the existence of such maps. Our main result moreover shows that, when the action $G \curvearrowright X$ is free, not only do such maps exist, but they are abundant.

In what follows, we call a subset $M \subseteq X$ **syndetic** if $X = F \cdot M$ for some finite $F \subseteq G$. Also, if μ is a Borel probability measure on X, then recall that the measure algebra MALG_μ is the collection of Borel subsets of X modulo μ-null sets. It is a Polish space under the metric $d([A]_\mu, [B]_\mu) = \mu(A \triangle B)$, where $[A]_\mu$ denotes the equivalence class of A in MALG_μ and $\triangle$ denotes symmetric difference.

Theorem 1.1. *Let $G \curvearrowright X$ be a free Borel action of G on a standard Borel space X. Then there exists a G-equivariant Borel map $f : X \to 2^G$ with $\overline{f(X)} \subseteq \mathrm{Free}(2^G)$. Furthermore:*

1. *Suppose that $Y \subseteq X$ is a Borel set such that $X \setminus Y$ is syndetic, and $\phi : Y \to 2$ is a Borel function. Then there exists a G-equivariant Borel map $f : X \to 2^G$ with $\overline{f(X)} \subseteq \mathrm{Free}(2^G)$ and $f(y)(1_G) = \phi(y)$ for all $y \in Y$.*
2. *Let Y and ϕ be as in part (1). Then there exists a family $\{f_w\}_{w \in 2^{\mathbb{N}}}$ of maps each satisfying the conclusion of part (1), and with the further property that*

$$\overline{f_w(X)} \cap \overline{f_z(X)} = \varnothing$$

   for all distinct $w, z \in 2^{\mathbb{N}}$. In addition, the map $(w, x) \mapsto f_w(x)$ is Borel, and for each fixed $x \in X$ the map $w \mapsto f_w(x)$ is continuous.
3. *For any G-quasi-invariant Borel probability measure μ on X, the set*

$$\{[A]_\mu : A \subseteq X \textit{ is Borel and } f_A(X) \subseteq \mathrm{Free}(2^G)\}$$

   is dense G_δ in MALG_μ.

In general the maps $f : X \to 2^G$ provided by the above theorem will not be injective. For example, if G is amenable (or more generally sofic) and $G \curvearrowright X$ admits an invariant Borel probability measure μ, then there cannot exist an equivariant injection into 2^G if the entropy of $G \curvearrowright (X, \mu)$ is greater than $\log(2)$. We mention, however, that a long standing open problem due to Weiss asks whether there is an equivariant injection $f : X \to k^G$ for some $k \in \mathbb{N}$ whenever $G \curvearrowright X$ does not admit any invariant Borel probability measure, see [15, p. 324] and [3, Problem 5.7]. Tserunyan [14] has shown that such an injection does exist whenever $G \curvearrowright X$ admits a σ-compact realization, although in general the problem remains open even in the case $G = \mathbb{Z}$.

Theorem 1.1 has a number of applications. For example, it implies that if the equivalence relation generated by $G \curvearrowright \mathrm{Free}(2^G)$ is treeable, then all equivalence relations induced by free Borel actions of G are