Annals of Pure and Applied Logic 167 (2016) 22–48

Contents lists available at ScienceDirect

Annals of Pure and Applied Logic

www.elsevier.com/locate/apal

Simplicity of the automorphism groups of some Hrushovski constructions

David M. Evans [a,*], Zaniar Ghadernezhad [b,1], Katrin Tent [c]

[a] *Department of Mathematics, Imperial College London, London SW7 2AZ, UK*
[b] *School of Mathematics, Institute for Research in Fundamental Sciences (IPM), P.O. Box 19395-5746, Tehran, Iran*
[c] *Mathematisches Institut, Universität Münster, Einsteinstrasse 62, 48149 Münster, Germany*

ARTICLE INFO

Article history:
Received 11 December 2013
Received in revised form 12 May 2014
Accepted 10 September 2015
Available online 1 October 2015

MSC:
03C15
20B07
20B27
03C98

Keywords:
Automorphism groups
Hrushovski constructions
Simple groups

ABSTRACT

We show that the automorphism groups of certain countable structures obtained using the Hrushovski amalgamation method are simple groups. The structures we consider are the 'uncollapsed' structures of infinite Morley rank obtained by the ab initio construction and the (unstable) $\aleph_0$-categorical pseudoplanes. The simplicity of the automorphism groups of these follows from results which generalize work of Lascar and of Tent and Ziegler.

1. Introduction

In this paper, we show that the automorphism groups of certain countable structures obtained using the Hrushovski amalgamation method are simple groups. This answers a question raised in [11] (Question (iii) of the Introduction there). The structures we consider are the 'uncollapsed' structures of infinite Morley rank obtained by the ab initio construction in [6] and the (unstable) $\aleph_0$-categorical pseudoplanes in [5]. The simplicity of the automorphism groups of these follows from some quite general results which should be of wider interest and applicability. Although much of the intuition (and some of the motivation) behind these results is model-theoretic, the paper requires no particular knowledge of model theory.

* Corresponding author.
E-mail addresses: david.evans@imperial.ac.uk (D.M. Evans), zaniar.gh@gmail.com (Z. Ghadernezhad), tent@math.uni-muenster.de (K. Tent).

[1] The author was supported by funding from the European Community's Seventh Framework Programme FP7/2007–2013 under grant agreement 23838.

http://dx.doi.org/10.1016/j.apal.2015.09.002

1.1. Background

The methods we use have their origins in the paper [10] of Lascar and it will be helpful to recall some of the results from there. Suppose M is a countable saturated structure with a $\emptyset$-definable strongly minimal subset D such that M is in the algebraic closure of D. Denote the dimension function on D coming from algebraic closure by dim. Consider $G = \mathrm{Aut}(M/\mathrm{acl}(\emptyset))$, the automorphisms of M which fix every element (of M^{eq}) algebraic over $\emptyset$. Suppose $g \in G$ is *unbounded* in the sense that for all $n \in \mathbb{N}$ there is a finite $X \subseteq D$ such that $\dim(gX/X) > n$. Then ([10], Théorème 2) the conjugacy class g^G generates G. In particular, if all non-identity elements of G are unbounded, then G is a simple group.

It is worth noting what Lascar's result says in the 'classical' cases where $M = D$. If M is a pure set, so G is the full symmetric group $\mathrm{Sym}(M)$, then $g \in G$ is bounded if and only if it is finitary. If M is a countably infinite dimensional vector space over a countable division ring F, then G is the general linear group $\mathrm{GL}(\aleph_0, F)$ and $g \in G$ is bounded if and only if it has an eigenspace of finite codimension. So in these cases, Lascar's result implies the well known results, due to Schreier and Ulam [13] in the case of the symmetric group, and due to Rosenberg [12] in the case of the general linear group, that G modulo the bounded part is simple. If M is an algebraically closed field of characteristic zero (and of countably infinite transcendence rank), then it can be shown that all non-identity automorphisms are unbounded, so in this case G is simple (note that $\mathrm{acl}(\emptyset)$ is the algebraic closure of the prime field).

Lascar's result is used directly in [4] to give examples of simple groups with a BN-pair which do not arise from algebraic groups. Ideas from Lascar's proof were used by Gardener [2] to give an analogue of Rosenberg's result for classical groups of countably infinite dimension over finite fields.

More recently, Lascar's ideas have been used in other contexts by Macpherson and Tent [11] and by Tent and Ziegler [16]. A key feature in both of these papers is the use of a natural independence relation or notion of free amalgamation on M. In [11], M is a homogeneous structure arising from a free amalgamation class of finite structures. Assuming $G = \mathrm{Aut}(M) \neq \mathrm{Sym}(M)$ is transitive on M, it is shown that G is simple. The free amalgamation here can be viewed as giving a notion of independence on M, and [16] formalizes this into the notion of a *stationary independence relation* on M ([16], Definition 2.1; cf. Definition 2.2 here). Generalising Lascar's notion of unboundedness, [16] introduces the notion of $g \in \mathrm{Aut}(M)$ *moving almost maximally* with respect to the independence relation (cf. Definition 2.6 here). It is shown ([16], Corollary 5.4) that in this case, every element of G is a product of 16 conjugates of g.

1.2. Main results

The paper contains two types of results. In Sections 2 and 3 we give general results along the lines of Lascar's result and the result of Tent and Ziegler; in Sections 4 and 5 we apply these to the Hrushovski constructions. We first describe our generalisations of the results of [10] and [16]. As these require a number of technical definitions, we shall not state the results precisely in this introduction.

In the results of [11] and [16], algebraic closure in M is trivial. In Section 2 here we adapt the results of [16] to remove this restriction. So M will be a countable structure, cl an $\mathrm{Aut}(M)$-invariant closure operation on M and we are interested in $G = \mathrm{Aut}(M/\mathrm{cl}(\emptyset))$. We define (Definition 2.2) the notion of a stationary independence relation *compatible with* cl and observe (Theorem 2.7) that the above result of Tent and Ziegler also holds in this wider context.

In Section 3, we assume that an integer-valued dimension function d gives the closure cl^d and the independence notion $\mathop{\smile\!\!\!\!\!\!\mid}^d$. This is the case in the Hrushovski construction which interests us, and of course is also the case in the almost strongly minimal situation of Lascar (where the closure is algebraic closure and dimension is given by Morley rank). We also assume a condition which we call *monodimensionality* (Definition 3.5) as a replacement for the assumption of almost strong minimality in Lascar's result. The main result here is Corollary 3.13: there is a natural notion of an automorphism being cl^d-bounded (Definition 3.12); such