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Model theoretic properties of the Urysohn sphere

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ABSTRACT

We characterize model theoretic properties of the Urysohn sphere as a metric structure in continuous logic. In particular, our first main result shows that the theory of the Urysohn sphere is SOP_n for all $n \geq 3$, but does not have the fully finite strong order property. Our second main result is a geometric characterization of dividing independence in the theory of the Urysohn sphere. We further show that this characterization satisfies the extension axiom, and so forking and dividing are the same for complete types. Our results require continuous analogs of several tools and notions in classification theory. While many of these results are undoubtedly known to researchers in the field, they have not previously appeared in publication. Therefore, we include a full exposition of these results for general continuous theories.

1. Introduction

The Urysohn sphere is the unique complete separable metric space of diameter 1, which is ultrahomogeneous and isometrically embeds every separable metric space of diameter ≤ 1. As a metric space, the Urysohn sphere is an important example in descriptive set theory, infinitary Ramsey theory, and topological dynamics of automorphism groups. As a model theoretic structure, the Urysohn sphere is most naturally studied by way of continuous logic. Indeed, the theory of the Urysohn sphere can be considered as the model completion of the "empty theory" in the continuous language containing only a predicate for the metric. As such, the Urysohn sphere is often used as a fundamental example of the kind of structure well-suited for study in continuous logic. Previous work on the model theory of the Urysohn sphere can be found in [7,8,15].

In [7], Goldbring and Ealy characterize thorn-forking in the Urysohn sphere and show that the theory is rosy (with respect to finitary imaginaries). They also include an argument, due to Pillay, that the Urysohn sphere is not simple, and mention unpublished observations of Berenstein and Usvyatsov that the Urysohn

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sphere is SOP_3, but without the strict order property. Altogether, this previous work motivates the following questions, which we answer in this paper.

1. How complicated is the theory of the Urysohn sphere with respect to commonly considered model theoretic dividing lines?
2. What is the nature of forking independence in this theory?

In particular, our first main result shows that the Urysohn sphere has the n-strong order property (SOP_n) for all $n \geq 3$, but does not have the fully finite strong order property. This significantly strengthens the previously known classification.[1] Moreover, answering a question of Starchenko, we show that the Urysohn sphere has the tree property of the second kind. In our second main result, we characterize forking and dividing for complete types in terms of basic distance calculations on metric spaces. A corollary of this characterization is that forking and dividing are the same for complete types, which indicates some good behavior of nonforking despite the model theoretic complexity of the theory. These results answer questions posed by Goldbring and Starchenko.

The previous results require an understanding of several notions concerning classification theory and forking and dividing. Therefore, this paper also serves to organize and verify continuous versions of several standard facts in this area.

First, we formulate continuous definitions of Shelah's strong order properties, and prove equivalent versions using amalgamation of indiscernible sequences. We then formulate definitions of forking and dividing for continuous logic, which are directly adapted from the discrete "syntactic" definitions. We also state equivalent versions of these definitions, which are familiar from sources in both discrete and continuous model theory. In Appendix A, we adapt the discrete proofs of these equivalences to continuous logic. In particular, we define dividing via k-inconsistency of infinite sequences and prove the equivalence of this definition with the more common version, found in [3], which uses inconsistency of indiscernible sequences. We also give a syntactic definition of forking resulting from a continuous version of "implying a disjunction of dividing formulas", and prove its equivalence with the definition motivated by the existence of nonforking extensions.

A common tool in the previous results is the ability to assume indiscernibility for infinite sequences, which witness certain model theoretic behavior. In discrete logic, this is done by taking indiscernible realizations of the *Ehrenfeucht–Mostowski type*. The existence of such realizations is a standard application of Ramsey's Theorem. In Appendix A, we define the EM-type for continuous logic, and prove that Ramsey's Theorem can be used to obtain indiscernible sequences realizing these types.

The outline of the paper is as follows. In Section 2 we translate Shelah's SOP_n-hierarchy to continuous logic, and prove that these dividing lines are detected by *n-cyclic* indiscernible sequences, i.e. indiscernible sequences whose 2-type can be consistently amalgamated in an n-cycle (see Definition 2.2). We then define forking and dividing and prove, for continuous logic, a standard fact that the equivalence of these notions is witnessed by the extension axiom for nondividing. Section 3 contains our main results concerning the theory of the Urysohn sphere, denoted $\mathcal{U}$. We first define $\mathcal{U}$, and recall the result, due to Henson, that $\mathrm{Th}(\mathcal{U})$ is separably categorical and eliminates quantifiers. We then give a classification of the model theoretic complexity of the Urysohn sphere. In particular, we show that $\mathrm{Th}(\mathcal{U})$ is SOP_n for all $n \geq 3$, but does not have the fully finite strong order property. We also prove that $\mathrm{Th}(\mathcal{U})$ has TP_2. We then turn our attention to nonforking independence. We first give a combinatorial characterization of dividing, which is formulated from basic distance calculations in metric spaces. We then show that this characterization satisfies the extension axiom, and so forking and dividing are the same for complete types. Using the characterization

[1] After obtaining our results, we later learned that the unpublished work of Berenstein and Usvyatsov, referenced in [7], also included a demonstration of SOP_n for general $n \geq 3$.