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Expressive completeness through logically tractable models

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ABSTRACT

How can we prove that some fragment of a given logic has the power to define precisely all structural properties that satisfy some characteristic semantic preservation condition? This issue is a fundamental one for classical model theory and applications in non-classical settings alike. While methods differ greatly, and while the classical methods can usually not be matched for instance in the setting of finite model theory, this note surveys some interesting commonality revolving around the use and availability of tractable representatives in the relevant model classes. The construction of models in which simple invariants like partial types based on some weak fragment control all the relevant structural properties, may be seen at the heart of such questions. We highlight some constructions involving degrees of acyclicity and saturation that can be achieved in finite model constructions, and discuss their uses towards expressive completeness w.r.t. bisimulation based equivalences in hypergraphs and relational structures. The emphasis is on the combinatorial challenges in such more constructive approaches that work in non-classical settings and especially in finite model theory. One new result concerns expressive completeness w.r.t. guarded negation bisimulation, a back-and-forth equivalence involving local homomorphisms.

1. Introduction

Model theory explores the logical definability of structural properties, i.e., the expressiveness of logics. Structural properties are here the material against which the semantics of logics under consideration is measured.

Classes of structural properties can be specified, for instance, in terms of other logics, but also in terms of a priori extra-logical considerations like algebraic or recursion-theoretic and algorithmic criteria. In a universal-algebraic tradition, closure conditions are a key example; in relation to the semantics of logical formalisms they are usually described as preservation conditions. In connection with algorithmic criteria, levels of computational complexity carve out natural classes of structural properties.

Logics are specified in terms of syntax and semantics. A logic may or may not provide the expressiveness to deal with certain classes of structural properties. The logics under consideration can be very concrete fragments of established systems, e.g., certain syntactic fragments of first-order logic if one is looking for

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the expressive means to capture just those first-order properties that satisfy some natural closure property. In different context, e.g., if one is looking for the expressive means to capture all structural properties of a given computational complexity, one may want to cast the net for candidate logics as wide as possible and restrict them by no more than the most rudimentary criteria concerning the manner in which their syntax and semantics are presented.

In any of these situations, precise matches between some class of structural properties and the expressiveness of some logic are particularly attractive. For the logic involved in such a match, one obtains a model-theoretic characterisation of its expressiveness, an answer to the question:

- Which properties precisely can be expressed?

For the class of structural properties involved, one obtains a descriptive characterisation:

- Syntax that captures these, and just these, properties.

Both aspects are of interest for the model-theoretic study of logical semantics. But typically they also have applications beyond the quintessentially model-theoretic interest. In particular, the availability of a logic that precisely captures a given class of properties provides a language for the specification and manipulation of just these properties, which is rich enough to express all intended properties and safeguarded against breaching the underlying semantic constraint. It may in addition provide a tool for surveying and for analysing this class of properties, including for instance the possibility to determine whether a given property is of this kind.

This paper concentrates on one technical aspect shared by various expressive completeness results, from typical classical examples to more recent explorations in finite model theory. The obvious differences have often been stressed: classical expressive completeness results for fragments of first-order logic are typically compactness based; alternative, more constructive and combinatorial arguments are required in those cases where expressive completeness results can be established in finite model theory. Meanwhile the growing number of qualified expressive completeness results in finite model theory [19,15–17,20,10,3,4,18] calls for a re-assessment of the earlier primarily negative view that focused on "failures in the finite" in comparison to the well-known classical preservation theorems. Also the major methodological differences may have hidden some interesting commonality that does prevail more often than had first been appreciated.

In this paper I attempt to describe one such common aspect that seems to link techniques across the divide of classical versus finite model theory, which I see in the use of logically tractable models. As a first approximation think of logical tractability w.r.t. some weaker logic L, whose expressive power is to be established, as a criterion that guarantees that, in certain well-behaved structures, L is unusually strong in the sense that L-descriptions of configurations (L-types) determine the behaviour of these configurations w.r.t. some stronger logic or up to some more powerful notion of equivalence than is usually associated with L.

Structures with saturation properties provide striking classical examples of this kind. Consider ω-saturated τ-structures $\mathfrak{A}$ and $\mathfrak{B}$ in some fixed finite relational vocabulary τ. If tuples $\mathbf{a} \in \mathfrak{A}$ and $\mathbf{b} \in \mathfrak{B}$ satisfy exactly the same first-order formulae $\varphi(\mathbf{x}) \in \mathrm{FO}[\tau]$, then, due to ω-saturation, $\mathbf{a}$ and $\mathbf{b}$ are linked by a back-and-forth system of local isomorphisms that establishes that $\mathfrak{A}, \mathbf{a}$ and $\mathfrak{B}, \mathbf{b}$ are related by *partial isomorphy*, $\mathfrak{A}, \mathbf{a} \simeq_{\mathrm{part}} \mathfrak{B}, \mathbf{b}$, and, by Karp's theorem, satisfy exactly the same formulae in the infinitary logic $\mathrm{FO}_\infty[\tau]$.[1] So the class of ω-saturated structures reduces FO_∞-equivalence (partial isomorphy) to FO-equivalence (elementary equivalence), and we may regard ω-saturated models as

[1] Here FO_∞ denotes the logic which allows for conjunctions and disjunctions over arbitrary sets of formulae; classical notation is $L_{\infty\omega}$ for FO_∞, in line with $L_{\omega\omega}$ for FO.