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An exposition of Hrushovski's New Strongly Minimal Set

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ABSTRACT

We give an exposition of Hrushovski's New Strongly Minimal Set (1993): A strongly minimal theory which is not locally modular but does not interpret an infinite field. We give an exposition of his construction.

In [5] E. Hrushovski proved the following theorem:

Theorem 0.1 *(Hrushovski's New Strongly Minimal Set). There is a strongly minimal theory which is not locally modular but does not interpret an infinite group.*

This refuted a conjecture of B. Zilber that a strongly minimal theory must either be locally modular or interpret an infinite field (see [7]). Hrushovski's method was extended and applied to many other questions, for example to the fusion of two strongly minimal theories [4] or recently to the construction of a bad field in [2].

There were also attempts to simplify Hrushovski's original constructions. For the fusion this was the content of [3]. I tried to give a short account of the New Strongly Minimal Set in a tutorial at the Barcelona Logic Colloquium 2011. The present article is a slightly expanded version of that talk.

1. Strongly minimal theories

An infinite L-structure M is *minimal* if every definable subset of M is either finite or cofinite. A complete L-theory T is *strongly minimal* if all its models are minimal. There are three typical examples:

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- Infinite sets without structure.
- Infinite vector spaces over a finite field.
- Algebraically closed fields.

The *algebraic closure* $\mathrm{acl}(A)$ of a subset A of M is the union of all finite A-definable subsets. In algebraically closed fields this coincides with the field-theoretic algebraic closure. In minimal structures acl has a special property:

Lemma 1.1. *In a minimal structure* acl *defines a pregeometry.*

A *pregeometry* (M, Cl) is a set M with an operator $Cl : \mathfrak{P}(M) \to \mathfrak{P}(M)$ such that for all $X, Y \subset M$ and $a, b \in M$

(a) $X \subset Cl(X)$, (Reflexivity)
(b) $X \subset Y \Rightarrow Cl(X) \subset Cl(Y)$, (Monotonicity)
(c) $Cl(Cl(X)) = Cl(X)$, (Transitivity)
(d) $a \in Cl(Xb) \setminus Cl(X) \Rightarrow b \in Cl(Xa)$, (Exchange)
(e) $Cl(X)$ is the union of all $Cl(A)$, (Finite character)

where A ranges over all finite subsets of X.

An operator with (a), (b) and (c) is called a closure operator. Note that (e) implies (b).

Proof of 1.1. All properties except Exchange are true in general and do not need the minimality of M. To prove the exchange property, assume $a \in \mathrm{acl}(Ab)$ and $b \notin \mathrm{acl}(Aa)$. There is a formula $\phi(x, y)$ with parameters in A such that $\phi(M, b)$ contains a and is finite, say with m elements. We can choose ϕ in such a way that $\phi(M, b')$ has at most m elements for all b'. Since b is not algebraic over Aa, $\phi(a, M)$ must be infinite. But M is minimal, so the complement $\neg\phi(a, M)$ is finite, say with n elements. Assume that there are pairwise different elements $a_0, \ldots, a_m$ such that each $\neg\phi(a_i, M)$ has at most n elements. Then for some b', $\phi(M, b')$ contains all the a_i, which contradicts the choice of ϕ. So there are at most m many a' such that $\neg\phi(a', M)$ has n elements. This shows that a is algebraic over A. □

Let X be a subset of M. A *basis* of X is a subset X_0 which *generates* X in the sense that $X \subset Cl(X_0)$ and is *independent*, which means that no element x of X_0 is in the closure $X_0 \setminus \{x\}$.

Lemma 1.2. *Every set X has a basis. All these bases have the same cardinality, the* dimension *of X.*

Proof. See [6, Lemma C 1.6]. □

In the three examples given above the dimension is computed as follows: If M is an infinite set without structure, the dimension of X is its cardinality. If M is an infinite vector space over a finite field, the dimension of a subset is the linear dimension of the subspace it generates. If M is an algebraically closed field, $\dim(X)$ is the transcendence degree of the subfield generated by X.

The dimension function, restricted to finite sets, has the following properties:

(1) $\dim(\emptyset) = 0$.
(2) $\dim(\{a\}) \leqslant 1$.
(3) $\dim(A \cup B) + \dim(A \cap B) \leqslant \dim(A) + \dim(B)$. (Submodularity)
(4) $A \subset B \Rightarrow \dim(A) \leqslant \dim(B)$. (Monotonicity)