Annals of Pure and Applied Logic 166 (2015) 342–357

Contents lists available at ScienceDirect

Annals of Pure and Applied Logic

www.elsevier.com/locate/apal

Superrosy fields and valuations

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ARTICLE INFO

Article history:
Received 15 August 2013
Received in revised form 23 July 2014
Accepted 23 July 2014
Available online 9 December 2014

MSC:
03C60
12J10
12L12
03C45

Keywords:
Superrosy field
Valuation

ABSTRACT

We prove that every non-trivial valuation on an infinite superrosy field of positive characteristic has divisible value group and algebraically closed residue field. In fact, we prove the following more general result. Let K be a field such that for every finite extension L of K and for every natural number $n > 0$ the index $[L^* : (L^*)^n]$ is finite and, if $\mathrm{char}(K) = p > 0$ and $f : L \to L$ is given by $f(x) = x^p - x$, the index $[L^+ : f[L]]$ is also finite. Then either there is a non-trivial definable valuation on K, or every non-trivial valuation on K has divisible value group and, if $\mathrm{char}(K) > 0$, it has algebraically closed residue field. In the zero characteristic case, we get some partial results of this kind.

We also notice that minimal fields have the property that every non-trivial valuation has divisible value group and algebraically closed residue field.

0. Introduction

A motivation for our work comes from some open structural questions concerning fields in various model-theoretic contexts.

A fundamental theorem says that each infinite superstable field is algebraically closed [18,4]. An important generalization of superstable theories is the class of supersimple theories and yet more general class of superrosy theories. Superrosy theories with NIP (the non-independence property) also form a generalization of superstable theories which is "orthogonal" to supersimple theories in the sense that each supersimple theory with NIP is superstable. Recall that a field K is bounded if for every natural number $n > 0$, K has only finitely many separable extensions of degree n (up to isomorphism over K); equivalently, the absolute Galois group of K is small, i.e., it has only finitely many closed subgroups of any finite index. It is known from [12] that perfect, bounded, PAC (pseudo algebraically closed) fields are supersimple. A well-known conjecture predicts the converse:

Conjecture 1. *Each infinite supersimple field is perfect, bounded and PAC.*

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[1] Research supported by NCN grant 2012/07/B/ST1/03513.

http://dx.doi.org/10.1016/j.apal.2014.11.006

A complementary conjecture on infinite superrosy fields with NIP was formulated in [6].

Conjecture 2. *Each infinite superrosy field with NIP is either algebraically or real closed.*

Recall that both algebraically closed and real closed fields are superrosy with NIP. After dropping the NIP assumption, one has to extend the list of possibilities in the conclusion of the above conjecture. Namely, since perfect, bounded, PAC fields as well as orderable, bounded, PRC (pseudo real closed) fields are known to be superrosy [20, Appendix A], the following conjecture is strongest possible. (See Section 4 for the definition of PRC fields, which is chosen so that PAC fields are PRC.)

Conjecture 3. *Each infinite superrosy field is perfect, bounded and PRC.*

It is known that a PAC field is simple if and only if it is bounded [12,2,3]; it is supersimple if and only if it is perfect and bounded. Similarly, a PRC field is superrosy if and only if it is perfect and bounded (see Fact 4.1). Thus, in Conjectures 1 and 3, once we know that the field is PAC [resp. PRC], the rest of the conclusion is automatically satisfied. It is also easy to see that Conjecture 3 implies Conjecture 1, because one can show that orderable PRC fields have strict order property, and so they are not simple (see Remark 4.2). By Fact 4.3, Conjecture 3 also implies Conjecture 2.

We will be often talking about definable valuations. Throughout the paper 'definable' means 'definable with parameters'.

There are interesting questions and conjectures concerning NIP fields (without assuming superrosiness). By [13], infinite NIP fields are closed under Artin–Schreier extensions. A. Hasson and S. Shelah formulated some dichotomies between nice algebraic properties of the field in question and the existence of non-trivial definable valuations. In particular, one can expect that the following is true.

Conjecture 4. *Suppose K is an infinite field with NIP with the property that for every finite extension L of K and for every natural number $n > 0$ the index $[L^* : (L^*)^n]$ is finite. Then either there is a non-trivial definable valuation on K, or K is either algebraically or real closed.*

Note that if a pure field K is algebraically or real closed, then there is no non-trivial definable valuation on K (e.g. because K is superrosy and we have Fact 1.8). Notice also that by Facts 1.8 and 1.9, Conjecture 4 implies Conjecture 2.

Another interesting problem is to classify strongly dependent fields [22, Section 5].

Independently of the questions of A. Hasson and S. Shelah in the NIP context, our approach to attack Conjectures 2 and 3 was to assume that the field in question does not satisfy the conclusion and try to produce a non-trivial definable valuation (existence of which contradicts rosiness by Fact 1.8). This approach led us to the following conjecture whose assumptions generalize the situations from Conjectures 1, 2, 3 and 4, but whose conclusion is weaker than the conclusions of these conjectures (see Section 4 for explanations). So, one could say that it is a common approximation of these conjectures. Before we formulate our conjecture, let us introduce a certain name for the fields satisfying its assumptions, which was suggested by the referee.

Definition 5. *We say that a field K is radically bounded if for every finite extension L of K and for any natural number $n > 0$ the index $[L^* : (L^*)^n]$ is finite and, if $\mathrm{char}(K) = p > 0$ and $f: L \to L$ is given by $f(x) = x^p - x$, then the index $[L^+ : f[L]]$ is also finite.*

It is a well-known fact that each perfect, bounded field is radically bounded, whereas the converse is not true as it has been very recently shown in [10]. Fact 1.9 tells us that superrosy fields are radically bounded. In fact, Remark 2.5 and the proof of [24, Theorem 5.6.5] show that they are perfect and bounded, but we will not need it.