Annals of Pure and Applied Logic 164 (2013) 251–283

Contents lists available at SciVerse ScienceDirect

Annals of Pure and Applied Logic

www.elsevier.com/locate/apal

Order algebraizable logics

J.G. Raftery

School of Mathematics, Statistics and Computer Science, University of KwaZulu-Natal, Private Bag X54001, Durban 4000, South Africa

ARTICLE INFO

Article history:
Received 16 October 2009
Received in revised form 28 September 2012
Accepted 10 October 2012
Available online 24 October 2012
Communicated by S.N. Artemov

MSC:
03G27

Keywords:
Consequence relation
Deductive system
Order algebraizable logic
Polarity
Quasi-order
Abstract algebraic logic

ABSTRACT

This paper develops an order-theoretic generalization of Blok and Pigozzi's notion of an algebraizable logic. Unavoidably, the ordered model class of a logic, when it exists, is not unique. For uniqueness, the definition must be relativized, either syntactically or semantically. In sentential systems, for instance, the order algebraization process may be required to respect a given but arbitrary *polarity* on the signature. With every deductive filter of an algebra of the pertinent type, the polarity associates a reflexive and transitive relation called a *Leibniz order*, analogous to the Leibniz congruence of abstract algebraic logic (AAL). Some core results of AAL are extended here to sentential systems with a polarity. In particular, such a system is order algebraizable if the Leibniz order operator has the following four independent properties: (i) it is injective, (ii) it is isotonic, (iii) it commutes with the inverse image operator of any algebraic homomorphism, and (iv) it produces anti-symmetric orders when applied to filters that define *reduced* matrix models. Conversely, if a sentential system is order algebraizable in some way, then the order algebraization process naturally induces a polarity for which the Leibniz order operator has properties (i)–(iv).

1. Introduction

1.1. Logic and algebra

Algebra, in its strict sense, concerns sets equipped with *operations* (and no relations); its atomic formulas are therefore *equations*. From this point of view, the demand that a logic possesses an *algebraic* semantics is nontrivial. Unlike the demand for a matrix semantics, it can fail—even in logics susceptible to the Lindenbaum–Tarski construction. Of course, numerous deductive systems do admit an algebraic semantics, but this on its own does not motivate the algebraic perspective in logic. Indeed, a familiar criticism of 'algebraic' logic comes to mind—that algebraic models and syntax are sometimes too similar for the former to throw really interesting light on the latter. The criticism has less sting when it comes to our understanding of *families* of logics, as opposed to isolated systems. Here, algebraic methods have been instrumental in providing deep insights.[1] In most such applications, the role of algebra is not confined to the provision of models. An implicit algebraic notion of *equivalence* for pairs of deductive systems is simultaneously at work, and it interacts with the formation of extensions, facilitating a large-scale transference of meta-logical data.

E-mail address: raftery@ukzn.ac.za.

[1] Among intermediate and modal logics, this is borne out, for instance, by the classification of systems with various interpolation or definability properties [23], the resolution of decision problems involving admissible rules [64], and the determination of degrees of incompleteness (see [14,37,62] and their references). More recently, for substructural logics, the persistence of properties like cut elimination has been illuminated by algebraic characterizations, advancing the traditionally non-algebraic field of proof theory [15]. Also, meta-logical demands sometimes reduce to algebraic properties of a *categorical* nature. Then, category equivalences in the algebraic domain become bridges for the immediate transfer of important information from one family of logics to another—perhaps in a different signature—in the absence of a direct syntactic translation (see [25,26] for some contemporary examples).

0168-0072/$ – see front matter
http://dx.doi.org/10.1016/j.apal.2012.10.013

In particular, if we take the last syllables of the word ‘algebraizable’ seriously, as Blok and Pigozzi [10] do, they ought to mean more than the possession of an algebraic semantics. Although soundness and completeness theorems (in two directions) are to be expected, they do not suffice for algebra*ization*. Nowadays, a deductive system is said to be *algebraizable* if it is fully *equivalent*—in a well-understood sense—to the *equational* consequence relation of a class of pure algebras (with a common signature). This concept has evolved from the analysis of [10], and the precise definition of ‘equivalent’ will be recalled in Section 4. Instead of focussing on Lindenbaum algebras, it associates with each deductive system an algebra of *theories* that is not merely a lattice; it captures the structure of substitution in the form of *operations* on theories, and it asks that the theory algebras of two equivalent systems be *isomorphic* in the usual algebraic sense. This allows us to forget such data as the *shapes* of formulas, while retaining a faithful picture of deductive relationships and the passage to extensions. By a remarkable result of abstract algebraic logic, recounted in Theorem 4.3, every such equivalence is induced by a well-behaved pair of syntactic translations.

Moreover, a purely algebraic invariant in the theory of equivalence—called the *Leibniz operator*—leads to a classification of *all* deductive systems, not only the algebraizable ones. It makes the falsification of properties like algebraizability more practical, and the resulting classes of systems are suitably stable (see [17,22,60]). Because equivalence and the Leibniz operator have nothing to do with any choice of semantics, they remind us that algebraic *methods* yield more than just algebraization.

1.2. Order

As it happens, many *non*-algebraizable logics still have a semantics consisting of algebras with a *partial order*, and order-theoretic analogues of algebraizability are of interest. The important concept of *interpolation* is an obvious motivating factor. Semantic accounts of *deductive* interpolation properties (applying to expressions like $\alpha \vdash \beta$) are already available in a setting that includes all algebraizable logics [19]. But *implicative* interpolation properties (for expressions like $\vdash \alpha \to \beta$) cannot be characterized to the same degree of generality, unless our semantic vocabulary includes a pertinent order relation.

In any departure from purely algebraic models, we must guard against generalizations that are ad hoc, unstable or too specialized. Because the notion of equivalence mentioned in 1.1 *is* purely algebraic, the following definition (essentially from [57]) recommends itself, and it is the main topic of this paper:

> a deductive system is *order algebraizable* if it is equivalent to the *inequational* consequence relation $\models_{\mathsf{K}}^{\leqslant}$ of a class K of partially ordered similar algebras.

In $\models_{\mathsf{K}}^{\leqslant}$, an inequation $\alpha \preccurlyeq \beta$ is regarded as a consequence of $\{\alpha_i \preccurlyeq \beta_i \colon\ i \in I\}$ iff the possibly infinite sentence

$$\forall \bar{x}\big(\big(\&_{i\in I}\alpha_i(\bar{x}) \leqslant \beta_i(\bar{x})\big) \Rightarrow \alpha(\bar{x}) \leqslant \beta(\bar{x})\big)$$

is true in K. Intrinsic characterizations of order algebraizability will be provided too, but the definition's appeal to a purely algebraic form of equivalence is a safeguard against idiosyncrasy.

1.3. Outline of results

A concise summary of this paper's results is presented here. Readers can alternatively skip to Section 2, as nothing in the sequel will depend on the present subsection.

Because the equality relation is a partial order, all algebraizable systems are order algebraizable. The Lambek calculus and the intensional fragments of linear and relevance logic are examples of order algebraizable systems that are not algebraizable. But the concept of order algebraization is not redundant in algebraizable logics (see below). Sequent calculi are often order algebraizable in a simple manner (see Theorem 5.7) and, partly for this reason, we shall be more concerned here with *sentential* logics, a.k.a. *Hilbert systems*.

Every order algebraizable sentential system is *equivalential*, i.e., something resembling a well-behaved bi-conditional ($\leftrightarrow$) is definable in the system. The converse is false, so the central concept of this paper is genuinely new. Indeed, certain fragments of Anderson and Belnap's *Entailment* logic **E** cannot be order algebraized in any way, despite being equivalential (Theorem 7.7). This follows from an intrinsic characterization of the order algebraizable sentential systems, viz. Theorem 7.1. The characterization in 7.1 is useful as a means of confirming order algebraizability, but it not readily falsifiable, because of its syntactic nature. It suffices for the aforementioned fragments of **E**, owing to the simplicity of their signatures. For richer equivalential systems, we cannot hope to disprove order algebraizability without first characterizing it intrinsically in model-theoretic terms. The ideal solution would be a characterization in terms of the Leibniz operator, analogous to the ones for algebraizability [10] and for other meta-logical properties [17,58]. But no useful characterization of this kind is known.

A related problem is that an order algebraizable system may be equivalent to multiple inequational consequence relations. This contrasts with the uniqueness of the equivalent model class for an algebraizable logic, which follows from special properties of the equality relation [10]. If we want a unique *ordered* model class, we must relativize the idea of order algebraization, either syntactically or semantically.