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Realizability models refuting Ishihara's boundedness principle

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ABSTRACT

Ishihara's *boundedness principle* BD-$\mathbb{N}$ was introduced in Ishihara (1992) [5] and has turned out to be most useful for constructive analysis, see e.g. Ishihara (2001) [6]. It is equivalent to the statement that every sequentially continuous function from $\mathbb{N}^{\mathbb{N}}$ to $\mathbb{N}$ is continuous w.r.t. the usual metric topology on $\mathbb{N}^{\mathbb{N}}$. We construct models for higher order arithmetic and intuitionistic set theory in which both every function from $\mathbb{N}^{\mathbb{N}}$ to $\mathbb{N}$ is sequentially continuous and in which the axiom of choice from $\mathbb{N}^{\mathbb{N}}$ to $\mathbb{N}$ holds. Since the latter is known to be inconsistent with the statement that all functions from $\mathbb{N}^{\mathbb{N}}$ to $\mathbb{N}$ are continuous these models refute BD-$\mathbb{N}$.

1. Introduction

In [5] H. Ishihara introduced the so-called *boundedness principle* BD-$\mathbb{N}$ which claims that every countable pseudobounded subset of $\mathbb{N}$ is bounded. Here $S \subseteq \mathbb{N}$ is called pseudobounded iff for every sequence $a \in S^{\mathbb{N}}$ there exists an $n \in \mathbb{N}$ such that $a_k < k$ for all $k \geqslant n$.[1] Obviously, the principle BD-$\mathbb{N}$ is classically valid. Moreover, it is a most useful amendment to Bishop style constructive mathematics in the sense that it is equivalent to a lot of useful mathematical theorems over a basic theory BISH of (predicative) constructive mathematics.[2] In [6] it is shown that BD-$\mathbb{N}$ is equivalent (over BISH) to each of the following prominent mathematical principles:

(1) Every sequentially continuous mapping from a complete separable metric (csm) space to a metric space is continuous.
(2) Banach's inverse mapping theorem.
(3) The open mapping theorem.

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[1] In [5] a subset S of $\mathbb{N}$ was called pseudobounded iff for every sequence (a_n) in S it holds that $\lim_{n\to\infty} \frac{a_n}{n} = 0$. But both notions of pseudoboundedness give rise to equivalent boundedness principles as shown in [6].

[2] As a codification of BISH one may take some variant of HA^{ω} or even P. Aczel's constructive set theory CZF, a predicative version of intuitionistic set theory IZF, together with number choice.

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(4) The closed graph theorem.
(5) The Banach–Steinhaus theorem.
(6) The sequential completeness of the space $\mathcal{D}$ of test functions (in the sense of L. Schwartz's theory of distributions).

In [3,13] it is shown that both Constructive Recursive Mathematics CRM and Brouwerian Intuitionism INT allow one to prove that all functions between complete separable metric spaces are continuous and thus, in particular, also BD-$\mathbb{N}$. Both CRM and INT are extensions of BISH postulating a classically inacceptable principle together with a classically valid principle stronger than BISH. From this point of view it "fits into the pattern" that BISH is in need of a further classically valid principle which exceeds basic constructivism (as represented e.g. by HA^ω or CZF) but which is still sufficiently constructive in nature. Ishihara's BD-$\mathbb{N}$ is a natural candidate for such a principle since it is equivalent to each of the most desirable principles (1)–(6) above and, moreover, constructively plausible since it holds both in

- number realizability combined with truth,
- function realizability combined with truth.

The reason is that number realizability validates CRM, function realizability validates INT and BD-$\mathbb{N}$ is classically valid and thus preserved when combining these realizability interpretations with truth (see e.g. [14]).

The aim of this note is to present in detail some very natural realizability models refuting BD-$\mathbb{N}$ but validating even intuitionistic Zermelo Fraenkel set theory IZF. These models have been sketched in Section 2.3 of the first author's PhD thesis [8]. The presentation there has been found moderately accessible by constructive mathematicians with little background in categorical logic. The current note is intended to make the result more widely accessible by reducing categorical logic to the bare minimum.

In the second section we observe that in presence of number choice $\mathsf{AC}_{0,0}$ a fairly weak continuity principle $\mathsf{CP}_0(\mathbb{N}^+)$ suffices to show (in BISH) that all functions from $\mathbb{N}^\mathbb{N}$ to $\mathbb{N}$ are sequentially continuous. In Section 3 we construct various realizability models which validate AC for finite types over $\mathbb{N}$ and $\mathsf{CP}_0(\mathbb{N}^+)$ but nevertheless refute Brouwer's continuity principle claiming that all functions from $\mathbb{N}^\mathbb{N}$ to $\mathbb{N}$ are continuous. Thus, these models will refute BD-$\mathbb{N}$ since it entails that all sequentially continuous functions from $\mathbb{N}^\mathbb{N}$ to $\mathbb{N}$ are continuous. We conclude in Section 4 with a discussion of related work.

2. Some theorems in constructive mathematics

Let $\mathbb{N}^+$ be the one point compactification of $\mathbb{N}$ consisting of all $\alpha \in 2^\mathbb{N}$ such that $n = m$ whenever $\alpha(n) = \alpha(m) = 1$. Obviously $\mathbb{N}^+$ is a retract of $2^\mathbb{N}$ and thus also of $\mathbb{N}^\mathbb{N}$. Let $\mathsf{CP}_0(\mathbb{N}^+)$ be the principle

$$\forall F\colon \mathbb{N}^{\mathbb{N}^+}.\exists n\colon \mathbb{N}.\forall \alpha\colon \mathbb{N}^+.\big(\forall k < n.\alpha(k) = 0\big) \to F(\alpha) = F\big(0^\infty\big)$$

where 0^∞ stands for the constant function with value 0. The following theorem is inspired by Proposition 4.4 of [1].

Theorem 2.1. *From $\mathsf{CP}_0(\mathbb{N}^+)$ it follows (in BISH) using number choice $\mathsf{AC}_{0,0}$ that all functions from $\mathbb{N}^\mathbb{N}$ to $\mathbb{N}$ are sequentially continuous.*

Proof. Suppose $F : \mathbb{N}^\mathbb{N} \to \mathbb{N}$. In order to show that F is sequentially continuous suppose (α_n) is a sequence in $\mathbb{N}^\mathbb{N}$ converging to β. We will show that $\lim_{n\to\infty} F(\alpha_n) = F(\beta)$.

First observe that by $\mathsf{AC}_{0,0}$ there exists $f \in \mathbb{N}^\mathbb{N}$ such that $\alpha_k(n) = \beta(n)$ whenever $k \geqslant f(n)$. Next we define a functional $H : \mathbb{N}^+ \times \mathbb{N} \to \mathbb{N}^\mathbb{N}$ as follows

$$H(\gamma, k)(n) = \begin{cases} \beta(n) & \text{if } \forall \ell < k.\gamma(\ell) = 0, \\ \alpha_m(n) & \text{if } m < k \text{ and } \gamma(m) = 1. \end{cases}$$

We will show now that there exists a functional $G : \mathbb{N}^+ \to \mathbb{N}^\mathbb{N}$ with $G(\gamma)(n) = \lim_{k\to\infty} H(\gamma, k)(n)$. Let $n \in \mathbb{N}$ and $k_0 = f(n)$. If there exists an $m < k_0$ with $\gamma(m) = 1$ then for all $k \geqslant k_0$ we have $H(\gamma, k)(n) = \alpha_m(n)$ and thus $\lim_{k\to\infty} H(\gamma, k)(n) = \alpha_m(n)$. On the other hand if $\forall \ell < k_0.\gamma(\ell) = 0$ then for all $k \geqslant k_0$ we have $H(\gamma, k)(n) = \beta(n)$ or $H(\gamma, k)(n) = \alpha_m(n)$ for some $m \geqslant k_0$ and thus also $H(\gamma, k)(n) = \alpha_m(n) = \beta(n)$ and accordingly $\lim_{k\to\infty} H(\gamma, k)(n) = \beta(n)$. Thus, we have shown that $\lim_{k\to\infty} H(\gamma, k) = G(\gamma)$ where

$$G(\gamma)(n) = \begin{cases} \beta(n) & \text{if } \forall \ell < f(n).\gamma(\ell) = 0, \\ \alpha_m(n) & \text{if } m < f(n) \text{ and } \gamma(m) = 1. \end{cases}$$

Obviously, we have $G(0^\infty) = \beta$ and $G(0^m10^\infty) = \alpha_m$.

Now applying assumption $\mathsf{CP}(\mathbb{N}^+)$ to the functional $F \circ G : \mathbb{N}^+ \to \mathbb{N}$ we obtain an $n \in \mathbb{N}$ such that $F(G(\gamma)) = F(G(0^\infty)) = F(\beta)$ whenever $\gamma(k) = 0$ for all $k < n$. Thus for all $m \geqslant n$ we have $F(\alpha_m) = F(G(0^m10^\infty)) = F(\beta)$, i.e. $\lim_{n\to\infty} F(\alpha_n) = F(\beta)$ as desired. □