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Abstract elementary classes and accessible categories

T. Beke [a,1], J. Rosický [b,*,2]

[a] Department of Mathematics, University of Massachusetts Lowell, One University Avenue, Lowell, MA 01854, USA
[b] Department of Mathematics and Statistics, Masaryk University, Faculty of Sciences, Kotlářská 2, 611 37 Brno, Czech Republic

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ABSTRACT

We investigate properties of accessible categories with directed colimits and their relationship with categories arising from Shelah's Abstract Elementary Classes. We also investigate ranks of objects in accessible categories, and the effect of accessible functors on ranks.

1. Introduction

M. Makkai and R. Paré [9] introduced accessible categories as categories sharing two typical properties of categories of structures described by infinitary first-order theories – the existence of sufficiently many directed colimits and the existence of a set of objects generating all objects by means of distinguished colimits. Their (purely category-theoretic) definition has since then found applications in various branches of mathematics. Very often, accessible categories have all directed colimits. These arise as categories of models and elementary embeddings of infinitary theories with finitary quantifiers. In model theory, S. Shelah went in a similar direction and introduced abstract elementary classes as a formalization of properties of models of generalized logics with finitary quantifiers. Our aim is to relate these two approaches. In Section 5, we introduce a hierarchy of accessible categories with directed colimits. The main result of that section, Corollary 5.7, sandwiches Shelah's Abstract Elementary Classes between two natural families of accessible categories.

Unlike abstract elementary classes, accessible categories are not equipped with canonical 'underlying sets'. Nonetheless, there exists a good substitute for 'size of the model', namely, the *presentability rank* of an object, that can be expressed purely in the language of category theory, i.e. in terms of objects and morphisms. In this sense, accessible categories take an extreme 'signature-free' and 'elements-free' view of abstract elementary classes. From this point of view, Shelah's Categoricity Conjecture, the driving force of abstract elementary classes (see [3]), turns on the subtle interaction between ranks of objects and directed colimits in accessible categories.

* Corresponding author.
E-mail addresses: tibor_beke@uml.edu (T. Beke), rosicky@math.muni.cz (J. Rosický).

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The connection between abstract elementary classes and accessible categories was discovered, independently, by M.J. Lieberman as well; see [7] and [8]. Our Corollary 5.7 simplifies Lieberman's description; cf. Proposition 4.8 and Claim 4.9 in [8].

2. Accessible categories

In order to define accessible categories one just needs the concept of a λ-directed colimit where λ is a regular cardinal number. This is a colimit over a diagram $D : \mathcal{D} \to \mathcal{K}$ where $\mathcal{D}$ is a λ-directed poset, considered as a category. An object K of a category $\mathcal{K}$ is called λ-*presentable* if its hom-functor $\hom(K, -) : \mathcal{K} \to \mathbf{Set}$ preserves λ-directed colimits; here **Set** is the category of sets.

A category $\mathcal{K}$ is called λ-*accessible*, where λ is a regular cardinal, provided that

(1) $\mathcal{K}$ has λ-directed colimits,
(2) $\mathcal{K}$ has a set $\mathcal{A}$ of λ-presentable objects such that every object of $\mathcal{K}$ is a λ-directed colimit of objects from $\mathcal{A}$.

A category is *accessible* if it is λ-accessible for some regular cardinal λ.

A *signature* Σ is a set of (infinitary) operation and relation symbols. These symbols are S-sorted where S is a set of sorts. It is advantageous to work with many-sorted signatures but it is easy to reduce them to single-sorted ones. One just replaces sorts by unary relation symbols and adds axioms saying that there are disjoint and cover the underlying set of a model. Thus the underlying set of an S-sorted structure A is the disjoint union of underlying sets A_s over all sorts $s \in S$. $|A|$ will denote the cardinality of the underlying set of the Σ-structure A. The category of all Σ-structures and homomorphisms (i.e., mappings preserving all operations and relations) is denoted by $\mathbf{Str}(\Sigma)$. A homomorphism is called a substructure embedding if it is injective and reflects all relations. Any inclusion of a substructure is a substructure embedding. Conversely, if $h : A \to B$ is a substructure embedding then A is isomorphic to the substructure $h(A)$ of B. The category of all Σ-structures and substructure embeddings is denoted by $\mathbf{Emb}(\Sigma)$. Both $\mathbf{Str}(\Sigma)$ and $\mathbf{Emb}(\Sigma)$ are accessible categories, cf. [2, 5.30 and 1.70].

A signature Σ is *finitary* if all relation and function symbols are finitary. For a finitary signature, the category $\mathbf{Str}(\Sigma)$ is locally finitely presentable and $\mathbf{Emb}(\Sigma)$ is finitely accessible. In both cases, there is a cardinal κ such that, for each regular cardinal $\kappa \leqslant \mu$, a Σ-structure K is μ-presentable if and only if $|K| < \mu$. This follows from the downward Löwenheim–Skolem theorem; see [9, 3.3.1].

Let λ be a cardinal and Σ be a λ-ary signature, i.e., all relation and function symbols have arity smaller than λ. Given a cardinal κ, the language $L_{\kappa\lambda}(\Sigma)$ allows less than κ-ary conjunctions and disjunctions and less than λ-ary quantifications. A substructure embedding of Σ-structures is called $L_{\kappa\lambda}$-*elementary* if it preserves all $L_{\kappa\lambda}$-formulas. A *theory* T is a set of sentences of $L_{\kappa\lambda}(\Sigma)$. $\mathbf{Mod}(T)$ denotes the category of T-models and homomorphisms, $\mathbf{Emb}(T)$ the category of T-models and substructure embeddings while $\mathbf{Elem}(T)$ will denote the category of T-models and $L_{\kappa\lambda}$-elementary embeddings. The category $\mathbf{Elem}(T)$ is accessible (see [2, 5.42]). For certain theories T, the category $\mathbf{Mod}(T)$ does not have μ-directed colimits for any regular cardinal μ and thus fails to be accessible.

A theory T is called *basic* if it consists of sentences

$$(\forall x)\big(\varphi(x) \Rightarrow \psi(x)\big)$$

where φ and ψ are positive-existential formulas and x is a string of variables. For a basic theory T, the category $\mathbf{Mod}(T)$ is accessible. Conversely, every accessible category is equivalent to the category of models and homomorphisms of a basic theory. All these facts can be found in [9] or [2].

Locally presentable categories are defined as cocomplete accessible categories. Following [2, 1.20], each locally λ-presentable category is locally μ-presentable for each regular cardinal $\mu > \lambda$. Let λ be an uncountable regular cardinal. The category $\mathbf{Pos}_\lambda$ of λ-directed posets and substructure embeddings is λ-accessible but it is not μ-accessible for all regular cardinals $\mu > \lambda$. Following [9, 2.3], let us write $\lambda \triangleleft \mu$ whenever $\mathbf{Pos}_\lambda$ is μ-accessible. There are arbitrarily large regular cardinals μ such that $\lambda \triangleleft \mu$ and, at the same time, arbitrarily large regular cardinals μ such that $\lambda \triangleleft \mu$ does not hold. Thus the *accessibility spectrum* of $\mathbf{Pos}_\lambda$ has a proper class of gaps. Generally, if $\lambda \triangleleft \mu$ then any λ-accessible category $\mathcal{K}$ is μ-accessible; see [9, 2.3.10] or [2, 2.11]. By [2, 2.13(1)], one has $\omega \triangleleft \lambda$ for every uncountable regular cardinal λ. Thus a finitely accessible category is μ-accessible for all uncountable regular cardinals μ.

Definition 2.1. We say that a category $\mathcal{K}$ is *well* λ-*accessible* if it is μ-accessible for each regular cardinal $\lambda \leqslant \mu$.
$\mathcal{K}$ is *well accessible* if it is well λ-accessible for some regular cardinal λ.

We have just seen that any locally presentable category and any finitely accessible category is well accessible.

Definition 2.2. Let λ be a regular cardinal. We say that an object K of a category $\mathcal{K}$ has *presentability rank* (or, for brevity, *rank*) λ if it is λ-presentable but not μ-presentable for any regular cardinal $\mu < \lambda$. We will write $\mathrm{rank}(K) = \lambda$.