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Hoop twist-structures

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ABSTRACT

In this paper, we introduce hoop twist-structure whose members are built as special squares of an arbitrary hoop. We show how our construction relates to eN4-lattices (N4-lattices) and implicative twist-structures. We prove that hoop twist-structures form a quasi-variety and characterize the AHT-congruences of each algebra in this quasi-variety in terms of the congruences of the associated hoop.

1. Introduction

Hoops were introduced in an unpublished manuscript by Büchi and Owens [7] in the mid of seventies, but their work is rich in ideas. The study of hoops is motivated by their occurrence both in universal algebra and algebraic logic.

Many of the familiar varieties of logic, such as modal algebras, cylindric algebras, relation algebras, Heyting algebras and Wajsberg algebras can be viewed as varieties of hoops with normal multiplicative operators.

Typical examples of hoops include both Brouwerian semilattices and Wajsberg hoops. The Brouwerian semilattices are the algebraic models of intuitionistic propositional logic. They are the $\{\wedge, \rightarrow, 1\}$-subreducts of Heyting algebras. Wajsberg hoops are the $\{*, \rightarrow, 1\}$-subreducts of Wajsberg algebras. Blok and Pigozzi [5] showed that bounded Wajsberg hoops are term-equivalent to Wajsberg algebras. Hence the variety of Wajsberg hoops is the algebraic semantics of the positive fragment of Lukasiewicz's infinite-valued logic. Thus the class of hoops is a natural common generalization of the varieties of Brouwerian semilattices and Wajsberg hoops.

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The twist-structure construction is used to study of algebras related to non-classical logics. The twist-structure construction has been applied to the solve logical and algebraic problems of algebras by using results on better known structures, such as Heyting or Boolean algebras. For example, Nelson lattices (the algebraic counterpart of Nelson logic [13]) can be represented as twist-structures over Heyting algebras [21]. Odinstov [16] introduced the algebraic models of paraconsistent Nelson logic [1] under the name of N4-lattices. These lattices can be represented by twist-structures of generalized Heyting algebras (also known as implicative lattices). The twist-structure construction has been used to study of residuated lattices [6,8] as an algebraic semantics for Paraconsistent Nelson's Logic [14,15,17,18]. Rivieccio [20] introduced the implicative twist-structures corresponding on a logical level, to the negation-implication fragment of the Arieli–Avron logic and, on an algebraic level, to the negation-implication subreducts of implicative bilattices [10,19].

The paper is organized as follows. In Section 2, some basic definitions and results are mentioned.

In Section 3, we use hoop to build a new algebra which we call hoop twist-structure and obtain some related results. We study the relation between hoop twist-structures and implicative twist-structures. We obtain condition that a hoop twist-structure is an eN4-lattice. Finally, we study the relation between hoop twist-structures and residuated lattices.

In Section 4, we introduce AHT-algebras as an abstract equational presentation for our twist-structures and obtain some of their properties. We prove that these abstract algebras correspond to the hoop twist-structures in Section 3.

Finally, in Section 5, we study the (AHT-)congruences on an AHT-algebra and characterize AHT-congruences in quasi-variety of AHT-algebras in terms of the congruences of its associated hoop.

2. Preliminaries

In this section, we include the basic definitions and some known results about hoops that we need in the rest of the paper. A thorough algebraic study of the class of the hoops may be found in [2–4] and [11,12].

Definition 2.1. ([7]) A hoop is an algebra $H = (H, *, \to, 1)$ such that $(H, *, 1)$ is a commutative monoid and for all $a, b, c \in H$,

(H1) $a \to a = 1$,
(H2) $a * (a \to b) = b * (b \to a)$,
(H3) $a \to (b \to c) = (a * b) \to c$.

Hence the class of all hoops is a variety. We denote the class of hoops by $\mathcal{HO}$. As we said in the introduction, hoops constitute the algebraic semantics for the $\bar{0}$-free fragments of the logics. In fact the variety of hoops is the equivalent algebraic semantics of the deductive system $S_{\mathcal{HO}}$, axiomatized by

(A1) $(p \to q) \to ((r \to p) \to (r \to q))$,
(A2) $(p \to (q \to r)) \to (q \to (p \to r))$,
(A3) $p \to (q \to p)$,
(A4) $p \to (q \to (p\&q))$,
(A5) $(p \to (q \to r)) \to ((p\&q) \to r)$,
(A6) $((p \to q)\&p) \to ((q \to p)\&q)$.

The only inference rule of $S_{\mathcal{HO}}$ is Modus Ponens:

$$\text{(MP)} \quad p, p \to q \vdash q.$$

If $H = (H, *, \to, 1)$ is a hoop, then the binary relation defined by $a \le b$ if and only if $a \to b = 1$ is a partial order on H. The underlying order in a hoop is always a semilattice order, where $a \wedge b = a * (a \to b)$; however it is not in general a lattice order.