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Strongly polynomial sequences as interpretations

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ABSTRACT

A strongly polynomial sequence of graphs (G_n) is a sequence $(G_n)_{n\in\mathbb{N}}$ of finite graphs such that, for every graph F, the number of homomorphisms from F to G_n is a fixed polynomial function of n (depending on F). For example, (K_n) is strongly polynomial since the number of homomorphisms from F to K_n is the chromatic polynomial of F evaluated at n. In earlier work of de la Harpe and Jaeger, and more recently of Averbouch, Garijo, Godlin, Goodall, Makowsky, Nešetřil, Tittmann, Zilber and others, various examples of strongly polynomial sequences and constructions for families of such sequences have been found, leading to analogues of the chromatic polynomial for fractional colourings and acyclic colourings, to choose two interesting examples.

We give a new model-theoretic method of constructing strongly polynomial sequences of graphs that uses interpretation schemes of graphs in more general relational structures. This surprisingly easy yet general method encompasses all previous constructions and produces many more. We conjecture that, under mild assumptions, all strongly polynomial sequences of graphs can be produced by the general method of quantifier-free interpretation of graphs in certain basic relational structures (essentially disjoint unions of transitive tournaments with added unary relations). We verify this conjecture for strongly polynomial sequences of graphs with uniformly bounded degree.

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1. Introduction

The chromatic polynomial $P(G, x)$ of a graph G, introduced by Birkhoff over a century ago, is such that for a positive integer n the value $P(G, n)$ is equal to the number of proper n-colourings of the graph G. Equivalently, $P(G, n)$ is the number $\hom(G, K_n)$ of homomorphisms from G to the complete graph K_n. It can thus be considered that the sequence $(K_n)_{n\in\mathbb{N}}$ defines the chromatic polynomial by means of homomorphism counting.

A *strongly polynomial sequence* of graphs is a sequence $(G_n)_{n\in\mathbb{N}}$ of finite graphs such that, for every graph F, the number of homomorphisms from F to G_n is a polynomial function of n (the polynomial depending on F and the sequence $(G_n)_{n\in\mathbb{N}}$, but not on n). A sequence $(G_n)_{n\in\mathbb{N}}$ of finite graphs is *polynomial* if this condition holds for sufficiently large $n \geq n_F$. The sequence of complete graphs (K_n) provides a classical example of a strongly polynomial sequence. A homomorphism from a graph F to a graph G is often called a *G-colouring of F*, the vertices of G being the "colours" assigned to vertices of F and the edges of G specifying the allowed colour combinations on the endpoints of an edge of F.

The notion of (strongly) polynomial sequences of graphs was introduced by de la Harpe and Jaeger [6] (as a generalization of the chromatic polynomial), in a paper which includes a characterization of polynomial sequences of graphs via (induced) subgraph counting and the construction of polynomial sequences by graph composition. The notion of a (strongly) polynomial sequence extends naturally to relational structures, thus allowing the use of standard yet powerful tools from model theory, like interpretations [7,9]. We use this to provide a construction which encompasses all previous constructions of strongly polynomial sequences and produces many more.

The "generalized colourings" introduced in [12,8] include only colourings invariant under all permutations of colours, which holds for K_n-colourings (that is, proper n-colourings), but not in general for G_n-colourings for other sequences of graphs $(G_n)_{n\in\mathbb{N}}$. Makowsky [11] moves towards a classification of polynomial graph invariants, but one that does not include the class of invariants we define in this paper. On the other hand, generalized colourings in the sense defined in [12,8] do include harmonious colourings (proper colourings with the further restriction that a given pair of colours appears only once on an edge) and others not expressible as the number of homomorphisms to terms of a graph sequence. Nevertheless, the formalism we introduce in this paper also extends to these types of colourings. We show that strongly polynomial sequences $(G_n)_{n\in\mathbb{N}}$ in the sense of de la Harpe and Jaeger (number of homomorphisms to G_n polynomial in n) have the further property that when imposing any condition on mappings from G to G_n that is expressible by a quantifier-free formula (such as being a homomorphism), the number of such mappings is again a polynomial in n (dependent only on the quantifier-free formula and on G). From this it is immediately seen that harmonious colourings and acyclic colourings, for example, are counted by polynomial functions just like the chromatic polynomial for usual proper colourings. (See Proposition 2.9, and its Corollary 2.10 and the paragraph that follows it.)

Garijo, Goodall and Nešetřil [4] give a construction of a broad class of strongly polynomial sequences by using representations of graphs by coloured rooted trees, which in particular incorporates the Tittmann–Averbouch–Makowsky polynomial [13] (not obtainable by graph composition and other operations known from [6] for building new polynomial sequences from old). In the language of this paper, this representation of graphs is an intepretation of graphs in coloured rooted trees and we thus find that the construction of [4] is a special instance of our method (see Section 5.1.6 below).

We extend the scope of the term "strongly polynomial" to sequences of general relational structures. The property of a sequence of relational structures being strongly polynomial is preserved under a rich variety of transformations afforded by the model-theoretic notion of an interpretation scheme. We start with "trivially" strongly polynomial sequences of relational structures, made from basic building blocks, and then by interpretation project these sequences onto graph sequences that are also strongly polynomial. The interpretation schemes that can be used here are wide-ranging (they need only be quantifier-free in their