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Intuitionistic common knowledge or belief

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A R T I C L E I N F O

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A B S T R A C T

Starting off from the usual language of modal logic for multi-agent systems dealing with the agents' knowledge/belief and common knowledge/belief we define so-called epistemic Kripke structures for intuitionistic (common) knowledge/belief. Then we introduce corresponding deductive systems and show that they are sound and complete with respect to these semantics.

1. Introduction

Common knowledge (and in particular the modal logic approach to common knowledge) has received a lot of attention in recent years; see, e.g., the textbooks Fagin, Halpern, Moses, and Vardi [4] and Meyer and van der Hoek [11] and the article [16] on common knowledge in the Stanford Encyclopedia of Philosophy by Sillari and Vanderschraaf. In these texts the general landscape of common knowledge is described and sound as well as complete formalizations of common knowledge in a multi-agent scenario are presented. Examples of proof-theoretic work on modal systems for common knowledge are Alberucci and Jäger [1], Brünnler and Studer [3], Jäger, Kretz, and Studer [7], Kretz and Studer [8], and Lescanne [9].

However, all these approaches are embedded in a framework of classical (multi-)modal logic. On the other hand, there is also the interesting – though not so popular – world of intuitionistic modal logics. First important results, including the completeness proof for the logic **IK**, are presented in Fischer Servi [5], and Simpson [15] provides an excellent survey of intuitionistic modal logics. It also leads to present research in this area.

In this article we start off from the traditional approach to common knowledge, but couched into an intuitionistic base logic. We present the system **ICK** for intuitionistic common knowledge and show that it is sound and complete. Hence this work is a technical contribution concerning an, as we think, natural system for dealing with common knowledge from an intuitionistic perspective.

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We do not enter into the discussion what the "right" intuitionistic epistemic logic is. There has been an interesting recent proposal by Artemov and Protopopescu in [2], but there are also alternative approaches by Williamson [17], Hirai [6], Proietti [14], and several others. This indicates that intuitionistic epistemic logic with and without common knowledge is an interesting area of ongoing research. But more work and a deeper conceptual analysis is necessary, and we hope that we will come back to this topic in a future publication.

Our formalism starts off from a framework for intuitionistic modal logic presented in Fischer Servi [5] and Plotkin and Stirling [13] and discussed from a broader perspective in Simpson [15]. We extend its $\Box$-fragment to several agents and treat common knowledge as a greatest fixed point, as it is common in epistemic logic; see, for example, Fagin, Halpern, Moses, and Vardi [4] or Meyer and van der Hoek [11]. More details about the relationship between our semantics and standard approaches in the literature are given at the end of Section 2.

The corresponding deductive systems are presented as sequent calculi, simply taking the intuitionistic variants of those in Alberucci and Jäger [1]. Their soundness with respect to our semantics will be obvious and their completeness will be shown in Section 4. There is nothing specific about this choice, the use of sequent calculi is a matter of personal taste rather than logical necessity. Equally well we could have adapted the Hilbert calculi of [5,13,15] to intuitionistic common knowledge.

2. The language $\mathcal{L}_{CK}$ and its semantics

In this section we introduce our language $\mathcal{L}_{CK}$ for intuitionistic common knowledge/belief and interpret its formulas over so-called epistemic Kripke structures, thus also providing a semantic approach to intuitionistic common knowledge/belief. The next section is dedicated to corresponding deductive systems.

The general assumption is that we want to deal with ℓ agents $a_1, \dots, a_\ell$. To formally express that agent a_i knows or believes α, we will write $\mathsf{K}_i(\alpha)$, and $\mathsf{C}(\alpha)$ says that α is common knowledge or common belief. Hence the language $\mathcal{L}_{CK}$ comprises the following primitive symbols:

- PS.1 Countably many atomic propositions p, q, r (possibly with subscripts); the collection of all atomic propositions is called *PROP*.
- PS.2 The logical constant $\bot$ and the logical connectives $\vee$, $\wedge$, and $\rightarrow$.
- PS.3 The modal operators $\mathsf{K}_1, \dots, \mathsf{K}_\ell, \mathsf{C}$.

The *formulas* α, β, γ, δ (possibly with subscripts) of $\mathcal{L}_{CK}$ are generated by the following BNF:

$$\alpha \ ::\equiv \ p \ | \ \bot \ | \ (\alpha \vee \alpha) \ | \ (\alpha \wedge \alpha) \ | \ (\alpha \rightarrow \alpha) \ | \ \mathsf{K}_i(\alpha) \ | \ \mathsf{C}(\alpha)$$

We make use of the standard syntactic abbreviations, for example, $\neg\alpha := (\alpha \rightarrow \bot)$ and $(\alpha \leftrightarrow \beta) := ((\alpha \rightarrow \beta) \wedge (\beta \rightarrow \alpha))$ and often omit parentheses and brackets if there is no danger of confusion. In addition, we abbreviate

$$\mathsf{E}(\alpha) \ := \ \mathsf{K}_1(\alpha) \wedge \dots \wedge \mathsf{K}_\ell(\alpha)$$

in order to express that "everybody knows α" or "everybody believes α", depending on what the $\mathsf{K}_i(\alpha)$ are supposed to formalize.

In the classical setting we have for all operators K_i the necessitation rule

$$\frac{\alpha}{\mathsf{K}_i(\alpha)}$$

and the normality axiom