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Equations for formally real meadows

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ABSTRACT

We consider the signatures $\Sigma_m = (0, 1, -, +, \cdot, ^{-1})$ of meadows and $(\Sigma_m, \mathbf{s})$ of signed meadows. We give two complete axiomatizations of the equational theories of the real numbers with respect to these signatures. In the first case, we extend the axiomatization of zero-totalized fields by a single axiom scheme expressing formal realness; the second axiomatization presupposes an ordering. We apply these completeness results in order to obtain complete axiomatizations of the complex numbers.

1. Introduction

The signature $\Sigma_f = (0, 1, -, +, \cdot)$ of *fields* has two constants 0 and 1, a unary function $-$, and two binary functions $+$ and $\cdot$. The first-order theory of fields is given by the axioms of commutative rings (see Table 1) and two additional axioms, namely

$$0 \neq 1,$$
$$x \neq 0 \rightarrow \exists y\ x \cdot y = 1.$$

A field F is said to be *ordered* if there exists a subset $F^{>0} \subseteq F$—the set of positive elements in F—such that $F^{>0}$ is closed under addition and multiplication, and F is the disjoint union of $F^{>0}$, $\{0\}$, and $\{-a \mid a \in F^{>0}\}$. Then F is totally ordered if we define $a > b$ to mean $a - b \in F^{>0}$. Moreover, if $a > b$, then $a + c > b + c$ for every c and $a \cdot c > b \cdot c$ for every $c \in F^{>0}$. The theory of ordered fields is formulated over the signature $\Sigma_{of} = (0, 1, -, +, \cdot, <)$. It has all the field axioms and, in addition, the axioms for a total ordering that is compatible with the field operations given in Table 2.

In 1927, the theory of ordered fields grew into the Artin–Schreier theory of ordered fields and formally real fields.

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Table 1
The set CR of axioms for commutative rings.

$$(x+y)+z = x+(y+z)$$
$$x+y = y+x$$
$$x+0 = x$$
$$x+(-x) = 0$$
$$(x\cdot y)\cdot z = x\cdot(y\cdot z)$$
$$x\cdot y = y\cdot x$$
$$1\cdot x = x$$
$$x\cdot(y+z) = x\cdot y + x\cdot z$$

Table 2
The set OF of axioms for ordered fields.

$$x \neq 0 \to (x < 0 \ \vee \ 0 < x) \tag{OF1}$$
$$x < y \to \neg(y < x \ \vee \ x = y) \tag{OF2}$$
$$x < y \to x + z < y + z \tag{OF3}$$
$$x < y \ \wedge \ 0 < z \to x\cdot z < y\cdot z \tag{OF4}$$

Definition 1.1. A field F is called *formally real* if -1 is not a sum of squares in F.

A main result of the Artin–Schreier theory (see e.g. [9]) states:

Proposition 1.2. *Let F be an arbitrary field. F is formally real if and only if for all $n \geq 0$ and all $x_0, \ldots, x_n \in F$ we have*

$$\sum_{i=0}^{n} x_i^2 = 0 \Rightarrow x_0 = \cdots = x_n = 0.$$

Formally real fields can therefore be axiomatized by the following infinite list of axioms, one for each $n \geq 0$,

$$\forall x_0 \forall x_1 \cdots \forall x_n \big(x_0\cdot x_0 + \cdots + x_n\cdot x_n = 0 \to (x_0 = 0 \ \wedge \ \cdots \ \wedge \ x_n = 0)\big).$$

A formally real field has no defined order relation. However, it is always possible to find an ordering (and often more) that will change a formally real field into an ordered field. One can view a formally real field as an ordered field where the ordering is not explicitly given. The fields of rational numbers $\mathbb{Q}$ and of real numbers $\mathbb{R}$ are examples.

Since the signature of fields does not include a multiplicative inverse, the axiom for the inverse is not universal, and therefore a substructure of a field closed under multiplication is not always a field. This can be remedied by adding a unary inverse operation $^{-1}$ to the language. In [6] *meadows* were defined as members of a variety specified by equations. A meadow is a commutative ring equipped with a total unary operation $^{-1}$ named *inverse* that satisfies $0^{-1} = 0$. Every field F can be expanded to a meadow (or *zero-totalized field*)