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A non-commutative generalization of Łukasiewicz rings

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ABSTRACT

The goal of the present article is to extend the study of commutative rings whose ideals form an MV-algebra as carried out by Belluce and Di Nola [1] to non-commutative rings. We study and characterize all rings whose ideals form a pseudo MV-algebra, which shall be called here generalized Łukasiewicz rings. We obtain that these are (up to isomorphism) exactly the direct sums of unitary special primary rings.

1. Introduction

A ring R is said to be generated by central idempotents, if for every $x \in R$, there exists a central idempotent element $e \in R$ such that $ex = x$. The (two-sided) ideals of such a ring R form a residuated lattice $A(R) := \langle \mathrm{Id}(R), \wedge, \vee, \odot, \rightarrow, \rightsquigarrow, \{0\}, R\rangle$, where

$$I \wedge J = I \cap J, I \vee J = I + J, I \odot J := I \cdot J,$$
$$I \rightarrow J := \{x \in R : Ix \subseteq J\}, I \rightsquigarrow J := \{x \in R : xI \subseteq J\}.$$

Of these rings, Belluce and Di Nola [1] investigated the commutative rings R for which $A(R)$ is an MV-algebra, which they called Łukasiewicz rings. Recall that MV-algebras, which constitute the algebraic counterpart of Łukasiewicz many value logic are categorically equivalent to Abelian ℓ-groups with strong units [4]. An important non-commutative generalization of MV-algebra, known as pseudo MV-algebra was introduced by Georgescu and Iorgulescu [8,9]. These have been studied extensively (see, e.g., [2,5–7]).

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The natural question that arises is what happens if one drops the commutativity assumption on Łukasiewicz rings. One would expect the residuated lattice $A(R)$ above to become a pseudo MV-algebra. One embarks then on the study of generalized Łukasiewicz rings (referred to in the article as GLRs, for short), which are rings (both commutative and non-commutative) for which $A(R)$ is a pseudo MV-algebra.

The main goal of this work is to completely characterize the rings R for which $A(R)$ is a pseudo MV-algebra. From the onset, the requirements on these rings appear to be very restrictive. However, it is quite remarkable that this class includes some very important classes of rings such as left Artinian chain rings, some special factors of Dubrovin valuation rings, and matrix rings over Łukasiewicz rings.

The paper is organized as follows. In section 2, we introduce and study generalized Łukasiewicz semi-rings, which comprised the Łukasiewicz semi-rings as studied in [1]. We show that these are dually equivalent to pseudo MV-algebras. In section 3, we introduce and study the main properties of generalized Łukasiewicz rings. In particular, we show that they are closed under finite direct products, quotients by ideals, and direct sums. In section 4, we prove a representation theorem for generalized Łukasiewicz rings. We obtain that (up to isomorphism), generalized Łukasiewicz rings are direct sums of unitary special primary rings.

In the paper, when the term ideal is used, it shall refer to two-sided ideal.

A pseudo-MV algebra can be defined as an algebra $A = \langle A, \oplus, ^-, ^\sim, 0, 1\rangle$ of type $(2, 1, 1, 0, 0)$ such that the following axioms hold for all $x, y, z \in A$ with an additional operation $x \odot y = (y^- \oplus x^-)^\sim$

(A1) $(x \oplus y) \oplus z = x \oplus (y \oplus z)$;
(A2) $x \oplus 0 = 0 \oplus x = x$;
(A3) $x \oplus 1 = 1 \oplus x = 1$;
(A4) $1^\sim = 0$, $1^- = 0$;
(A5) $(x^- \oplus y^-)^\sim = (x^\sim \oplus y^\sim)^-$;
(A6) $x \oplus (x^\sim \odot y) = y \oplus (y^\sim \odot x) = (x \odot y^-) \oplus y = (y \odot x^-) \oplus x$;
(A7) $x \odot (x^- \oplus y) = (x \oplus y^\sim) \odot y$;
(A8) $(x^-)^\sim = x$.

Every pseudo MV-algebra has an underline distributive lattice structure, where the order $\le$ is defined by:

$$x \le y \quad \text{if and only if} \quad x^- \oplus y = 1.$$

Moreover, the infimum and supremum in this order are given by:

$$\text{(i)}\quad x \vee y = x \oplus (x^\sim \odot y) = y \oplus (y^\sim \odot x) = (x \odot y^-) \oplus y = (y \odot x^-) \oplus x,$$
$$\text{(ii)}\quad x \wedge y = x \odot (x^- \oplus y) = y \odot (y^- \oplus x) = (x \oplus y^\sim) \odot y = (y \oplus x^\sim) \odot x.$$

The prototype of pseudo MV-algebra can be constructed from an ℓ-group as follows. Let G be an ℓ-group and u a positive element in G, then $\langle \Gamma(G, u), \oplus, ^-, ^\sim, 0, u\rangle$, where

$$\Gamma(G, u) := \{x \in G : 0 \le x \le u\},$$
$$x \oplus y := (x + y) \wedge u, \qquad x^\sim := -x + u,$$
$$x \odot y = (x - u + y) \vee 0, \qquad x^- := u - x,$$

is a pseudo MV-algebra.

In fact, a remarkable result due to Dvurečenskij [6] asserts that every pseudo MV-algebra is isomorphic to a pseudo MV-algebra of the form $\langle \Gamma(G, u), \oplus, ^-, ^\sim, 0, u\rangle$. Pseudo MV-algebras have a wealth of properties