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Reasoning about negligibility and proximity in the set of all hyperreals

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A B S T R A C T

We consider the binary relations of negligibility, comparability and proximity in the set of all hyperreals. Associating with negligibility, comparability and proximity the binary predicates N, C and P and the connectives $[N]$, $[C]$ and $[P]$, we consider a first-order theory based on these predicates and a modal logic based on these connectives. We investigate the axiomatization/completeness and the decidability/complexity of this first-order theory and this modal logic.

1. Introduction

Within the context of the modeling of the behavior of a complex system, when numeric information is useless or when available information is unprecise, the use of qualitative reasoning is often required [20,22]. It is a fact that engineering practice usually induces the experts to handle the symbols $\ll$ (“is negligible with respect to”) and $\simeq$ (“is in the proximity of”) while simplifying complex equations. Nevertheless, this rule of thumb has to be formalized if one intends to mechanically reproduce by means of algorithms the engineers ability to reason about the behavior of a complex system. This formalization task is at the heart of the qualitative reasoning enterprise.

Restricting his discussion to the relative orders of magnitude paradigm, Raiman [19] introduced a formal system, FOG, based on the binary relations Ne (“is negligible with respect to”), Co (“is comparable to”) and Vo (“is in the proximity of”). Without studying its completeness, he justified the use of FOG by showing the soundness of the inference rules of FOG with respect to nonstandard analysis, i.e. by interpreting Ne, Co and Vo as follows: $Ne(a,b)$ iff a/b is infinitesimal, $Co(a,b)$ iff a/b is appreciable and $Vo(a,b)$ iff $a/b-1$ is infinitesimal for each hyperreals a, b.

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Variants of *FOG* have been later introduced in order, for example, to incorporate numeric information [10] or to relate together different types of order-of-magnitude knowledge [21]. See also [11,17]. Nevertheless, there is something wrong with them: if the soundness or the complexity of the proposed formal systems are sometimes examined, their completeness with respect to such-and-such semantics is never studied. The first purpose of the present paper is to investigate the axiomatization/completeness and the decidability/complexity of the first-order theory the binary relations of negligibility, comparability and proximity give rise to in the set of all positive hyperreals.

Recently, modal languages for qualitative order-of-magnitude reasoning have been considered [4,6]. See also [5,16]. In these modal languages, connectives are associated to the binary relations of negligibility and comparability. Nevertheless, the first-order conditions put on these binary relations in the Kripke frames used to interpret these modal languages do not constitute a complete axiomatization of their first-order theory in the set of all positive hyperreals. The second purpose of the present paper is to investigate the axiomatization/completeness and the decidability/complexity of the modal logic the binary relations of negligibility, comparability and proximity give rise to in the set of all positive hyperreals.

The binary relations of negligibility, comparability and proximity in the set of all positive hyperreals will be presented in section 2. Section 3 will associate with negligibility, comparability and proximity the binary predicates N, C and P and will study the first-order theory based on these predicates. Section 4 will associate with negligibility, comparability and proximity the connectives $[N]$, $[C]$ and $[P]$ and will study the modal logic based on these connectives. Variants of our first-order and modal languages based, in the set of all positive hyperreals, on the relation of precedence and the operation of addition will be presented in section 5.

2. Hyperreals

2.1. What are the hyperreals?

In the set of all reals, there are no such things as infinitely small and infinitely large numbers. While reals all belong to the same order of magnitude, it is the fact that hyperreals are either infinitesimal, appreciable or unlimited which sets them apart. The thing is that hyperreals contains the reals as a subset, but also contains infinitely small (infinitesimal) numbers and infinitely large (unlimited) numbers. In mathematics, these new entities offer new definitions of familiar concepts like convergence and continuity [15]. In other areas of science and technology, they justify the algebraic processing of small numbers and large numbers that researchers and engineers often do—witness their use in multifarious domains like market models [9] for modeling option pricing and in electrical networks [23] for modeling infinite networks. In computer science and artificial intelligence, infinitesimal numbers and unlimited numbers have been used for analyzing texts [2] and reasoning about time in deductive databases [14].

2.2. Ultrapower construction of the hyperreals

Following the introduction to non-standard analysis proposed in [15], let us introduce a number of basic concepts. Let I be the set of all positive integers. We use $\mathbb{R}^I$ to denote the set of all real-valued sequences, $\mathcal{P}(I)$ to denote the power set of I and $\mathcal{P}(\mathcal{P}(I))$ to denote the power set of $\mathcal{P}(I)$. For a start, suppose that the notion of a large set of positive integers, in a sense that is to be determined, is at our disposal. Given $\mathbf{a}, \mathbf{b} \in \mathbb{R}^I$, we shall say that $\mathbf{a}$ agrees with $\mathbf{b}$ iff $\{n \in I\colon \mathbf{a}(n) = \mathbf{b}(n)\}$ is large. The set $\{n \in I\colon \mathbf{a}(n) = \mathbf{b}(n)\}$ may be thought of as a measure of the extent to which the statement "$\mathbf{a}$ agrees with $\mathbf{b}$" is true. In order to ensure that agreement between real-valued sequences is a non-trivial equivalence relation, the following conditions must be satisfied: