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Transformation of fractions into simple fractions in divisive meadows

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ABSTRACT

Meadows are alternatives for fields with a purely equational axiomatization. At the basis of meadows lies the decision to make the multiplicative inverse operation total by imposing that the multiplicative inverse of zero is zero. Divisive meadows are meadows with the multiplicative inverse operation replaced by a division operation. Viewing a fraction as a term over the signature of divisive meadows that is of the form p / q, we investigate which divisive meadows admit transformation of fractions into simple fractions, i.e. fractions without proper subterms that are fractions.

1. Introduction

To our knowledge, all existing definitions of a fraction are insufficiently precise to allow the validity of many non-trivial statements about fractions to be established. The work presented in this paper is concerned with the rigorous definition of a fraction and the validity of statements related to the question whether each fraction can be transformed into a simple fraction (colloquially described as a fraction where neither the numerator nor the denominator contains a fraction). This work is carried out in the setting of divisive meadows.

Because fields do not have a purely equational axiomatization, the axioms of a field cannot be used in applications of the theory of abstract data types to number systems based on rational, real or complex numbers. In [7], meadows are proposed as alternatives for fields with a purely equational axiomatization. At the basis of meadows lies the decision to make the multiplicative inverse operation total by imposing that the multiplicative inverse of zero is zero. A meadow is a commutative ring with a multiplicative identity element and a total multiplicative inverse operation satisfying the two equations $(x^{-1})^{-1} = x$ and $x \cdot (x \cdot x^{-1}) = x$. It follows from the axioms of a meadow that the multiplicative inverse operation also satisfies the equation

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$0^{-1} = 0$. All fields in which the multiplicative inverse of zero is zero, called zero-totalized fields, are meadows, but not conversely. Because of their purely equational axiomatization, all meadows are total algebras and the class of all meadows is a variety.

In [4], divisive meadows are proposed. A divisive meadow is a commutative ring with a multiplicative identity element and a total division operation satisfying the three equations $1 / (1 / x) = x$, $(x \cdot x) / x = x$, and $x / y = x \cdot (1 / y)$. It follows from the axioms of a divisive meadow that the division operation also satisfies the equation $x / 0 = 0$. We coined the alternative name inversive meadow for a meadow. The equational axiomatizations of inversive meadows and divisive meadows are essentially the same in the sense that they are definitionally equivalent.

We expect the zero-totalized multiplicative inverse and division operations of inversive and divisive meadows, which are conceptually and technically simpler than the conventional partial multiplicative inverse and division operations, to be useful in among other things mathematics education. We further believe that viewing fractions as terms over the signature of divisive meadows whose outermost operator is the division operator gives a rigorous definition of a fraction that can serve as a basis of a workable theory about fractions for teaching purposes at all levels of education (cf. [1]). Divisive meadows are more convenient than inversive meadows for the definition of fractions because, unlike the signature of inversive meadows, the signature of divisive meadows includes the division operator.

Viewing fractions as described above has two salient consequences: (i) fractions may contain variables and (ii) fractions may be interpreted in different divisive meadows. These consequences lead to the need to make a distinction between fractions and closed fractions and to consider properties of fractions relative to a particular divisive meadow. Viewing fractions as described above, many properties of fractions considered in the past turn out to be properties of closed fractions and/or properties of fractions relative to the divisive meadow of rational numbers.

For example, it is known from earlier work on meadows that closed fractions can be transformed into simple fractions, i.e. fractions without proper subterms that are fractions, if fractions are interpreted in the divisive meadow of rational numbers. Now the question arises whether the restriction to closed fractions can be dropped and whether this result goes through if fractions are interpreted in divisive meadows different from the divisive meadow of rational numbers.

In this paper, we investigate which divisive meadows admit transformation into simple fractions (for both the general case and the case of closed fractions). Some exemplary results are: (i) every model of the axioms of divisive meadow with a finite carrier admits transformation into simple fractions; (ii) every minimal model of the axioms of a divisive meadow with an infinite carrier does not admit transformation into simple fractions; (iii) the divisive meadow of rational numbers is the only minimal model of the axioms of a divisive meadow with an infinite carrier that admits transformation into simple fractions for closed fractions.

This paper is organized as follows. First, we give a survey of inversive meadows and divisive meadows which includes the signature and axioms for them, general results about them, and terminology used in the setting of meadows (Section 2). Next, we give the definitions concerning fractions and polynomials on which subsequent sections are based (Section 3) and establish some auxiliary results concerning divisive meadows which will be used in subsequent sections (Section 4). Then, we establish results about the transformation into simple fractions (Sections 5 and 6). Following this, we establish results that are related to the results in the preceding sections, but do not concern fractions (Section 7). Finally, we make some concluding remarks (Section 8).

We conclude this introduction with a corrective note on a remark made in [4] and later papers on meadows. Skew meadows, which differ from meadows only in that their multiplication is not required to be commutative, were already studied in [16,17], where they go by the name of desirable pseudo-fields. In 2009, we first read about desirable pseudo-fields in [19] and reported on it in [4]. However, we thought incorrectly