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Revisiting da Costa logic

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ABSTRACT

In [25] Priest developed the da Costa logic (**daC**); this is a paraconsistent logic which is also a co-intuitionistic logic that contains the logic $\mathbf{C}_\omega$. Due to its interesting properties it has been studied by Castiglioni, Ertola and Ferguson, and some remarkable results about it and its extensions are shown in [8,11]. In the present article we continue the study of **daC**, we prove that a restricted Hilbert system for **daC**, named DC, satisfies certain properties that help us show that this logic is not a maximal paraconsistent system. We also study an extension of **daC** called $\mathbf{PH_1}$ and we give different characterizations of it. Finally we compare **daC** and $\mathbf{PH_1}$ with several paraconsistent logics.

1. Introduction

Paraconsistent logic is a well-known topic in the scientific community, and there is a lot of work done in this area. Today, paraconsistent logic touches various topics, such as ontology, the philosophy of science, applied science and technology. Thus, any advance in this area will be welcomed [3].

Briefly speaking, following Jean-Yves Béziau [5], a logic is paraconsistent if it has a negation ($\neg$) which is paraconsistent in the sense that the relation $\alpha, \neg\alpha \vdash \beta$ does not always hold for arbitrary formulas α and β, and at the same time it has strong properties that justify calling it a negation. Nevertheless, there is no paraconsistent logic that is unanimously recognized as a "good one" [5], and there are different proposals on what a paraconsistent logic should be [7].

Along this article we will focus on two paraconsistent logics, one is da Costa logic (**daC**) [25], also known as Priest–da Costa logic, and the other one is an extension of **daC** known as $\mathbf{PH_1}$ [11]. We will discuss

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some relevant properties that can be considered as "desirable" in such logics, (particularly when our aim is to apply them to intelligent agents) and the relation of **daC** and $\mathbf{PH_1}$ with respect to some paraconsistent logics. The present work is mainly a theoretical contribution to the area of paraconsistent logic and is also related to modal and constructive logics. To some extent, our work is related to the area of Non-Monotonic Reasoning (NMR): several authors have applied different logics in the modeling of non-monotonic reasoning by means of completions [10,14]. In fact, in [18] an interesting approach for Knowledge Representation (KR) was proposed and developed in [17,24]. This approach can be supported by any paraconsistent logic stronger than or equal to $\mathbf{C}_\omega$ [22,23], which is the case of **daC** and $\mathbf{PH_1}$.

The structure of this article is as follows. In Section 2, we present the necessary background in order to simplify the reading of the article, and a brief introduction to several of the logics that are relevant for the results of later sections. In Section 3, we show that **daC** does not satisfy a number of intuitive properties concerning the behavior of negation, it is not a maximal paraconsistent logic in the strong sense, and we compare it with some paraconsistent logics.

In Section 4 we obtain different characterizations of $\mathbf{PH_1}$ and we compare it with some paraconsistent logics as well.

In Section 5, we present our conclusions, some conjectures and ideas for future work.

2. Background

We first introduce a few basic definitions and clarify concepts such as maximality. Then we define some properties of a negation and, finally, we give a brief review of some paraconsistent logics. We assume that the reader has some familiarity with basic results from logic such as chapter one in [15].

2.1. Logic systems

We consider a formal (propositional) language $\mathcal{L}$ built from: an enumerable set of atoms (denoted by $p,\ q,\ r,\ldots$) and the set of connectives $\{\wedge, \vee, \rightarrow, \neg\}$.[1] *Formulas* are constructed as usual and will be denoted as lowercase Greek letters. *Theories* are sets of formulas and will be denoted as uppercase Greek letters.

A *logic* is simply a set of formulas that is closed under Modus Ponens (*MP*) and substitution. All the logics involved in this article are propositional logics and all of them have the same language, $\mathcal{L}$.[2] The elements of a logic are called *theorems*. Let X be a logic, the notation $\vdash_X \alpha$ is used to state that the formula α is a theorem of X (i.e. $\alpha \in X$).[3] Thought many different approaches have been used to define logics, we will use two of them, axiomatic systems (also known as Hilbert style proof systems) and many-valued systems.

We say that a logic X is *weaker than or equal to* a logic Y if $X \subseteq Y$. Sometimes we refer to this as Y *extends* X. Similarly, we say that X is *stronger than or equal to* Y if $Y \subseteq X$.

2.2. Maximality properties

Maximality is a desirable property of paraconsistent logics. However, the notion of maximality in the field of paraconsistent logic is far from being unique. One can define maximality of a logic with respect to some other logic, as in [6]. But in the literature one can find stronger notions of maximality. For example, a notion of absolute maximality, in the sense that it is not defined with respect to some other given logic, is the following:

[1] We use $\alpha \leftrightarrow \beta$ to abbreviate $(\alpha \rightarrow \beta) \wedge (\beta \rightarrow \alpha)$.

[2] In the following sections we will mention some connectives and constants that do not appear in $\mathcal{L}$. All such cases will be abbreviations of formulas in the original language.

[3] We drop the subscript X in $\vdash_X$ when the given logic is understood from the context.