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Reflecting rules: A note on generalizing the deduction theorem

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ABSTRACT

The purpose of this brief note is to prove a limitative theorem for a generalization of the deduction theorem. I discuss the relationship between the deduction theorem and rules of inference. Often when the deduction theorem is claimed to fail, particularly in the case of normal modal logics, it is the result of a confusion over what the deduction theorem is trying to show. The classic deduction theorem is trying to show that all so-called 'derivable rules' can be encoded into the object language using the material conditional. The deduction theorem can be generalized in the sense that one can attempt to encode all types of rules into the object language. When a rule is encoded in this way I say that it is *reflected* in the object language. What I show, however, is that certain logics which reflect a certain kind of rule must be trivial. Therefore, my generalization of the deduction theorem does fail where the classic deduction theorem didn't.

1. Introduction

It is an often discussed question whether the deduction theorem fails for modal logics. One of the simple resolutions to this question is to say that it depends on how deduction or derivability—and so the deduction theorem—is construed. A quick way to see two different ways of construing deduction centers around the rule of necessitation.

Usually the deduction theorem is construed as: if $\Gamma, \beta \vdash \alpha$, then $\Gamma \vdash \beta \supset \alpha$. In this case, $\Gamma \vdash \alpha$ is generally interpreted as: the conclusion α (single formula) is derivable from the premises Γ (set of formulas). But the rule of necessitation, viz. that $\Box\alpha$ is derivable from α, is a standard rule of so-called normal modal logics. But generally, $\nvdash \alpha \supset \Box\alpha$, i.e., not all such conditionals are theorems. The failure of the deduction theorem results from using the same sense of 'derivable' in the rule of necessitation and for the turnstile in $\Gamma \vdash \alpha$. In $\Gamma \vdash \alpha$ rules for deduction can be applied to the elements of Γ to get α. In the case of necessitation,

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the rule can only be applied to *theorems*. So $\Box\alpha$ is derivable from α, but only when α is a theorem. For a contemporary and complete discussion of the deduction theorem, one can consult Hakli and Negri [8].

Distinct notions of derivability were unavailable to early discussions of modal logic, i.e., Lewis [11]. As Parry [14] points out, Lewis intended his system[2] of strict implication to be 'one whose "$p \prec q$" best represents "q is deducible from p" as ordinarily used' (p. 134). Thus the senses of 'derivable' were limited. The deduction theorem for strict implication would be important, then, since it should serve to translate all valid cases of 'q is deducible from p as ordinarily used' in to a theorem of strict implication, i.e., $p \prec q$. Although there were early results clarifying the correctness of the deduction theorem for modal logic, i.e., Barcan [1], the full range of possible senses of 'derivability' was not investigated until much later. Modern studies of inference rules in proof-theory have introduced a number of distinctions that help clarify the notion of derivability, see Fagin et al. [4] and Iemhoff and Metcalfe [9].

In what follows I will fix some concepts and notation, then rehearse some ways of making sense of *derivability* via inference rules. The goal is to prove a limitative theorem for a general conception of the deduction theorem which I call the 'Failure of Reflection Theorem'.

2. Rules in logic

2.1. Logics

Before getting to rules, I have to say something about logics and how they are specified. First, each logic L is specified by its consequence relation $\Vdash_L$, that is, a logic is a relation between sets of formulas and a single formula. The formulas are specified as part of a recursively constructed propositional language $\mathcal{L}$, built up from a set of atomic formulas $\mathbf{At} = \{\, \mathbf{p}, \mathbf{q}, \dots \}$. In the interests of generality, the specific structure of the language is left open.

As per usual, there are two ways of generating a logic: by proof or by truth. When we give a proof theory for a logic, we then have to specify the conditions under which a proof (be it in a sequent system or a Hilbert-style system) validates a claim of consequence $\Gamma \Vdash_L \alpha$. In sequent systems, it is done by saying that the sequent $\Gamma \Rightarrow \alpha$ is producible from the rules for manipulating sequents. In the case of a Hilbert-style system, usually, $\Gamma \Vdash_L \alpha$ is validated iff there are $\gamma_1, \dots, \gamma_n \in \Gamma$ such that $(\gamma_1 \wedge \dots \gamma_n) \supset \alpha$ is a theorem of the logic L, i.e., $(\gamma_1 \wedge \dots \gamma_n) \supset \alpha$ is the last formula in a sequence of formulas in which each formula is either an axiom or producible from formulas earlier in the sequence using the rules of the system. Of course that requires a language with conjunction and a conditional. I will leave further discussion of proof-theory until later.

A logic L is specified by a semantics when there is a set of structures $\mathcal{S}$, and a relation of satisfaction ($\vDash$) between the structures of $\mathcal{S}$ and $\mathcal{L}$ which says when a formula of $\mathcal{L}$ is true at a structure $\mathcal{M} \in \mathcal{S}$. Then, $\Gamma \Vdash_L \alpha$ iff for all $\mathcal{M} \in \mathcal{S}$, if $\mathcal{M} \vDash \gamma$ for all $\gamma \in \Gamma$, then $\mathcal{M} \vDash \alpha$. Some examples of this: classical logic can be specified by the set of truth valuations: $v : \mathbf{At} \to \{\, T, F \,\}$, where $v \vDash \alpha$ iff $v^*(\alpha) = T$ where v^* is the unique extension of v to the whole of $\mathcal{L}$ for classical logic. Similarly, for modal logics: A formula α is satisfied in a structure $\langle \mathcal{M}, w \rangle$ which is an ordered pair of a model $\mathcal{M} = \langle W, R, v \rangle$ (in which W is a non-empty set also referred to as $|\mathcal{M}|$, $R \subseteq W \times W$, with $v : \mathbf{At} \to \mathcal{P}(W)$), and $w \in |\mathcal{M}|$. The set of structures for a modal logic can then be specified by a set of models M by setting $\mathcal{S} = \{\, \langle \mathcal{M}, w \rangle : w \in |\mathcal{M}| \;\&\; \mathcal{M} \in M \,\}$. In the case of the modal logic K, M is the set of all models, in the case of S4, it can be specified by setting M to be all models in which the relation R on $|\mathcal{M}|$ is reflexive and transitive. These few examples will suffice for the moment.

In order to display whether the consequence relation of a logic is specified by a proof theory or semantics we shall use $\Gamma \vdash_L \alpha$ for proof theory, and $\Gamma \vDash_L \alpha$ for semantics. Accordingly, completeness proofs are

[2] The system Lewis thought was *the* system of strict implication is his S2, see Parry's article for more specifics