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INFINITELY MANY SOLUTIONS FOR AN ELLIPTIC PROBLEM INVOLVING CRITICAL NONLINEARITY*

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Abstract We study the following elliptic problem:

$$\begin{cases} -\mathrm{div}(a(x)Du) = Q(x)|u|^{2^*-2}u + \lambda u & x \in \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

Under certain assumptions on a and Q, we obtain existence of infinitely many solutions by variational method.

Key words semilinear elliptic equations; infinitely many solutions; variational method

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1 Introduction

Let $N \geq 3$, $2^* = \frac{2N}{N-2}$, and Ω be an open bounded domain in $\mathbb{R}^N$. We consider the following elliptic problem:

$$\begin{cases} -\mathrm{div}(a(x)Du) = Q(x)|u|^{2^*-2}u + \lambda u & x \in \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $a,\ Q \in C^4(\bar{\Omega})$, $a(x) \geq a_0 > 0$, $Q(x) \geq Q_0 > 0$, and $\lambda > 0$ is a positive constant.

The functional corresponding to (1.1) is

$$I(u) = \frac{1}{2}\int_\Omega \big(a(x)|Du|^2 - \lambda u^2\big)\mathrm{d}x - \frac{1}{2^*}\int_\Omega Q(x)|u|^{2^*}\mathrm{d}x, \quad u \in H_0^1(\Omega). \tag{1.2}$$

Since the embedding of $H_0^1(\Omega)$ into $L^{2^*}(\Omega)$ is not compact, the functional $I(u)$ does not satisfies the Palais-Smale condition ((PS) condition for short). This loss of compactness creates a lot of difficulties when variational method is used to obtain the existence result for (1.1).

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Let us first recall some well known results for problem (1.1) when $a(x) \equiv 1$ and $Q(x) \equiv 1$. By using the Pohozaev identity [17], problem (1.1) has no nontrivial solution if $\lambda \le 0$ and Ω is star-shaped. On the other hand, Brezis and Nirenberg [6] proved that if $N \ge 4$ and $\lambda \in (0, \lambda_1)$, where λ_1 is the first eigenvalue of $-\Delta$ in $H_0^1(\Omega)$, (1.1) has a positive solution, while it was proved by Capozzi, Fortunato and Palmieri [9], Ambrosetti and Struwe [2] that (1.1) has a nontrivial solution if $N \ge 4$ and $\lambda > 0$. Concerning the multiplicity results for (1.1), Cerami, Solimini and Struwe [10] showed that (1.1) has a pair of sign-changing solutions if $N \ge 4$ and $\lambda \in (0, \lambda_1)$, and (1.1) has infinitely many radial solutions if $N \ge 7$ and Ω is a ball. On the other hand, D.Fortunato and E.Jannelli showed in [15] that, for any real positive parameter λ and for all bounded domains Ω, which have suitable symmetry properties, (1.1) has infinitely many solutions when $N \ge 4$, while for $N = 3$, the number of solutions increases with λ. Recently, Devillanova and Solimini proved [11] that (1.1) has infinitely many solutions if $N \ge 7$ and $\lambda > 0$. In the lower dimensional cases $N = 4, 5, 6$, they also proved [12] that (1.1) has more than one pair of sign-changing solutions, if $\lambda \in (0, \lambda_1)$.

When one of the functions $a(x)$ and $Q(x)$ is not constant, it is difficult to obtain a sign-changing solution for (1.1) by using a variational method, because $I(u)$ does not satisfy $(\mathrm{PS})_c$ condition for any c larger than the smallest number, where the (PS) condition fails. The aim of this paper is to prove that (1.1) has infinitely many solutions if $N \ge 7, a(x)$ and $Q(x)$ satisfy some degenerate conditions near their critical points.

Since the functional $I(u)$ does not satisfy the (PS) condition, we first look at the following perturbed problem:

$$\begin{cases} -\mathrm{div}(a(x)Du) = Q(x)|u|^{2^*-2-\varepsilon}u + \lambda u & x \in \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.3}$$

where $\varepsilon > 0$ is a small constant.

The functional corresponding to (1.3) becomes

$$I_\varepsilon(u) = \frac{1}{2}\int_\Omega \big(a(x)|Du|^2 - \lambda u^2\big)\mathrm{d}x - \frac{1}{2^* - \varepsilon}\int_\Omega Q(x)|u|^{2^*-\varepsilon}\mathrm{d}x, \quad u \in H_0^1(\Omega). \tag{1.4}$$

Now $I_\varepsilon(u)$ is an even functional and satisfies the (PS) condition. So from [1, 19] (1.3) has infinitely many solutions. More precisely, there are positive numbers $c_{\varepsilon,l}$, $l = 1, 2\cdots$, with $c_{\varepsilon,l} \to +\infty$ as $l \to +\infty$, and a solution $u_{\varepsilon,l}$ for (1.3), satisfying

$$I_\varepsilon(u_{\varepsilon,l}) = c_{\varepsilon,l}.$$

Moreover, $c_{\varepsilon,l} \to c_l < +\infty$ as $\varepsilon \to 0$.

Now we want to study the behavior of $u_{\varepsilon,l}$ as $\varepsilon \to 0$ for each fixed l. If we can prove that under suitable assumptions on a and Q, $u_{\varepsilon,l}$ converges strongly in $H_0^1(\Omega)$ to u_l as $\varepsilon \to 0$, then u_l is a solution of (1.1) with $I(u_l) = c_l$. This will imply that (1.1) has infinitely many solutions. Before we can give the precise conditions for a and Q, we need to introduce some notation.

Define

$$\Sigma(x) = \frac{a^{N/2}(x)}{Q^{(N-2)/2}(x)}. \tag{1.5}$$

Let S be the set of all the critical points of $\Sigma(x)$. Let $\langle x, y\rangle$ denote the inner product of $x, y \in \mathbb{R}^N$.