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Stochastic completeness for graphs with curvature dimension conditions

Bobo Hua [a,b], Yong Lin [c,*]

[a] *School of Mathematical Sciences, LMNS, Fudan University, Shanghai 200433, China*
[b] *Shanghai Center for Mathematical Sciences, Fudan University, Shanghai 200433, China*
[c] *Department of Mathematics, Information School, Renmin University of China, Beijing 100872, China*

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A B S T R A C T

We prove pointwise gradient bounds for heat semigroups associated to general (possibly unbounded) Laplacians on infinite graphs satisfying the curvature dimension condition $\mathrm{CD}(K,\infty)$. Using gradient bounds, we show stochastic completeness for graphs satisfying the curvature dimension condition.

1. Introduction and main results

Let M be a complete, noncompact Riemannian manifold without boundary. It is called stochastically complete if

$$\int_M p_t(x,y)d\mathrm{vol}(y) = 1, \quad \forall t>0,\ x\in M, \tag{1}$$

* Corresponding author.
E-mail addresses: bobohua@fudan.edu.cn (B. Hua), linyong01@ruc.edu.cn (Y. Lin).

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where $p_t(\cdot,\cdot)$ is the (minimal) heat kernel on M. Yau [41] first proved that any complete Riemannian manifold with a uniform lower bound of Ricci curvature is stochastically complete. Karp and Li [23] showed the stochastic completeness in terms of the following volume growth property:

$$\mathrm{vol}(B_r(x)) \le Ce^{cr^2}, \quad \text{some } x \in M,\ \forall r > 0, \tag{2}$$

where $\mathrm{vol}(B_r(x))$ is the volume of the geodesic ball of radius r and centered at x. Varopoulos [35], Li [27] and Hsu [19] extended Yau's result to Riemannian manifolds with general conditions on Ricci curvature. So far, the optimal volume growth condition for stochastic completeness was given by Grigor'yan [14]. We refer to [15] for the literature on stochastic completeness of Riemannian manifolds. These results have been generalized to a quite general setting, namely, regular strongly local Dirichlet forms by Sturm [34].

Compared to local operators, graphs (discrete metric measure spaces) are nonlocal in nature and can be regarded as regular Dirichlet forms associated to jump processes. A general Markov semigroup is called a diffusion semigroup if chain rules hold for the associated infinitesimal generator, see Bakry, Gentil and Ledoux [3, Definition 1.11.1], which is a property related to the locality of the generator. As a common point of view to many graph analysts, the absence of chain rules for discrete Laplacians is the main difficulty for the analysis on graphs. This causes many problems and various interesting phenomena emerge on graphs. A graph is called *stochastically complete* (or conservative) if an equation similar to (1) holds for the continuous time heat kernel, see Definition 3.1. The stochastic completeness of graphs has been thoroughly studied by many authors [7,8,13,21,24–26,32,36–39]. In particular, the volume criterion (2) with respect to the graph distance is no longer true for unbounded Laplacians on graphs, see [39]. This can be circumvented by using intrinsic metrics introduced by Frank, Lenz and Wingert [9], see e.g. [11,13,22].

Gradient bounds of heat semigroups can be used to prove stochastic completeness. Nowadays, the so-called Γ-calculus has been well developed in the framework of general Markov semigroups where Γ is called the "carré du champ" operator, see [3, Definition 1.4.2]. Given a smooth function f on a Riemannian manifold, $\Gamma(f)$ stands for $|\nabla f|^2$, see Section 2 for the definition on graphs. Heuristically, on a Riemannian manifold M if one can show the gradient bound for the heat semigroup

$$\Gamma(P_t f) \le C_t P_t(\Gamma(f)), \quad \forall f \in C_0^\infty(M), \tag{3}$$

where $P_t = e^{t\Delta_M}$ is the heat semigroup induced by the Laplace–Beltrami operator Δ_M, C_t a function on t and $C_0^\infty(M)$ the space of compactly supported smooth functions on M, then the stochastic completeness follows from approximating the constant function $\mathbb{1}$ by compactly supported smooth functions. The gradient bounds (3) can be proved under curvature assumptions, e.g. a uniform lower bound of Ricci curvature, and then