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Zeta-polynomials for modular form periods ☆

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A B S T R A C T

Answering problems of Manin, we use the critical L-values of even weight $k \geq 4$ newforms $f \in S_k(\Gamma_0(N))$ to define zeta-polynomials $Z_f(s)$ which satisfy the functional equation $Z_f(s) = \pm Z_f(1-s)$, and which obey the Riemann Hypothesis: if $Z_f(\rho) = 0$, then $\mathrm{Re}(\rho) = 1/2$. The zeros of the $Z_f(s)$ on the critical line in t-aspect are distributed in a manner which is somewhat analogous to those of classical zeta-functions. These polynomials are assembled using (signed) Stirling numbers and "weighted moments" of critical L-values. In analogy with Ehrhart polynomials which keep track of integer points in polytopes, the $Z_f(s)$ encode arithmetic information. Assuming the Bloch–Kato Tamagawa Number Conjecture, they encode the arithmetic of a combinatorial arithmetic–geometric object which we call the "Bloch–Kato complex" for f. Loosely speaking, these are graded sums of weighted moments of

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orders of Šafarevič–Tate groups associated to the Tate twists of the modular motives.

1. Introduction and statement of results

Let $f \in S_k(\Gamma_0(N))$ be a newform of even weight k and level N. Associated to f is its L-function $L(f,s)$, which may be normalized so that the completed L-function

$$\Lambda(f,s) := \left(\frac{\sqrt{N}}{2\pi}\right)^s \Gamma(s)L(f,s),$$

satisfies the functional equation $\Lambda(f,s) = \epsilon(f)\Lambda(f,k-s)$, with $\epsilon(f) = \pm 1$. The *critical L-values* are the complex numbers $L(f,1)$, $L(f,2)$, ..., $L(f,k-1)$.

In a recent paper [14], Manin speculated on the existence of natural *zeta-polynomials* which can be canonically assembled from these critical values. A polynomial $Z(s)$ is a *zeta-polynomial* if it is arithmetic–geometric in origin, satisfies a functional equation of the form

$$Z(s) = \pm Z(1-s)$$

and obeys the Riemann Hypothesis: if $Z(\rho) = 0$, then $\mathrm{Re}(\rho) = 1/2$.

Here we confirm his speculation. To this end, we define the m-th *weighted moments* of critical values

$$M_f(m) := \sum_{j=0}^{k-2} \left(\frac{\sqrt{N}}{2\pi}\right)^{j+1} \frac{L(f,j+1)}{(k-2-j)!} j^m = \frac{1}{(k-2)!}\sum_{j=0}^{k-2}\binom{k-2}{j}\Lambda(f,j+1)j^m. \quad (1.1)$$

For positive integers n, we recall the usual generating function for the (signed) Stirling numbers of the first kind

$$(x)_n = x(x-1)(x-2)\cdots(x-n+1) =: \sum_{m=0}^{n} s(n,m)x^m. \quad (1.2)$$

Using these numbers we define the zeta-polynomial for these weighted moments by

$$Z_f(s) := \epsilon(f) \cdot \sum_{h=0}^{k-2} (-s)^h \sum_{m=0}^{k-2-h} \binom{m+h}{h} \cdot s(k-2, m+h) \cdot M_f(m). \quad (1.3)$$

To be a zeta-polynomial in the sense of Manin [14], we must show that $Z_f(s)$ satisfies a functional equation and the Riemann Hypothesis. Our first result confirms these properties.