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Chow ring of generically twisted varieties of complete flags ☆

Nikita A. Karpenko

Mathematical & Statistical Sciences, University of Alberta, Edmonton, Canada

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ABSTRACT

Let G be a split simple affine algebraic group of type A or C over a field k, and let E be a standard generic G-torsor over a field extension of k. We compute the Chow ring of the *variety of Borel subgroups of G* (also called the *variety of complete flags of G*), twisted by E. In most cases, the answer contains a large finite torsion subgroup. The torsion-free cases have been treated in the predecessor *Chow ring of some generically twisted flag varieties* by the author.

Contents

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E-mail address: karpenko@ualberta.ca.
URL: http://www.ualberta.ca/~karpenko.

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1. Introduction

Let G be a split semisimple affine algebraic group over a field k and let $\mathcal{B}$ be its *variety of Borel subgroups* (also called the *variety of complete flags* of G). A rational point on $\mathcal{B}$ is given by a Borel subgroup $B \subset G$. Picking up such a point, one establishes an isomorphism of $\mathcal{B}$ with the quotient variety G/B. The Chow ring $\operatorname{CH}\mathcal{B}$ of $\mathcal{B}$ is well-studied and understood.

The situation changes dramatically when we twist $\mathcal{B}$ by a G-torsor (i.e., a principle homogeneous space) E over a field extension $F \supset k$. The F-variety Y thus obtained is isomorphic to the quotient variety E/B. It is not reasonable to hope to understand $\operatorname{CH} Y$ in general.

If the torsor E is split, the situation is easy because $Y \simeq \mathcal{B}_F$. Whatever viewpoint is applied to measure how far is a given torsor E from the split one, the *standard generic torsors* are always among the winners. A standard generic torsor E is, by definition, the generic fiber of the quotient map $\operatorname{GL}_N \to \operatorname{GL}_N /G$ for an imbedding of G into the general linear group GL_N for some $N \geq 1$. In particular, E is a G-torsor over the function field

$$F := k(\operatorname{GL}_N /G) \supset k.$$

The Chow ring of the corresponding F-variety Y does not depend on the choice of the imbedding $G \hookrightarrow \operatorname{GL}_N$ and is the main object of study in the paper.

A standard tool of studying $\operatorname{CH} Y$ for an arbitrary smooth variety Y is the canonical epimorphism $\operatorname{CH} Y \to GK(Y)$, where $K(Y)$ is the Grothendieck ring of Y endowed with the *topological filtration* (also called *geometrical filtration* as well as *filtration by codimension of support*) and $GK(Y)$ is the associated graded ring. The kernel of this epimorphism is contained in the torsion subgroup $\operatorname{Tors}\operatorname{CH} X$ of $\operatorname{CH} X$ [6, Example 15.3.6] and is controlled by differentials of the Brown–Gersten–Quillen spectral sequence [16].

For our particular Y, the ring $\operatorname{CH} Y$ is generated by its component $\operatorname{CH}^1 Y$ [12, Example 2.4]. As a consequence, the topological filtration for our Y coincides with the gamma filtration [9, Remark 2.17]. The latter is the filtration generated by K-theoretical Chern classes, introduced by A. Grothendieck for arbitrary smooth varieties as an (at least theoretically) computable approximation of the topological filtration.

Here is the short version of the main result of the paper (see §4 and §5 for the proof):

Theorem 1.1. *Let G be a split simple affine algebraic group of type* A *or* C *and let Y be its variety of complete flags twisted by a standard generic torsor. Then the canonical epimorphism $\operatorname{CH} Y \to GK(Y)$ is an isomorphism.*