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Cluster structures on strata of flag varieties

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In memory of Andrei Zelevinsky, whose work has been a constant source of inspiration.

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ABSTRACT

We introduce some new Frobenius subcategories of the module category of a preprojective algebra of Dynkin type, and we show that they have a cluster structure in the sense of Buan–Iyama–Reiten–Scott. These categorical cluster structures yield cluster algebra structures in the coordinate rings of intersections of opposite Schubert cells.

1. Introduction

1.1. Let G be a simple and simply connected algebraic group over $\mathbb{C}$. We assume that G is simply-laced, that is, G is of type A, D, E in the Cartan–Killing classification. We fix a maximal torus H in G, a Borel subgroup B containing H, and we denote by B^- the Borel subgroup opposite to B with respect to H. Let $W = \mathrm{Norm}_G(H)/H$ be the Weyl group, with length function $w \mapsto \ell(w)$ and longest element w_0.

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We consider the flag variety $X = B^-\backslash G$, and we denote by $\pi : G \to X$ the natural projection $\pi(g) := B^-g$. The Bruhat decomposition

$$G = \bigsqcup_{w\in W} B^-wB^-$$

projects to the Schubert decomposition

$$X = \bigsqcup_{w\in W} C_w, \tag{1}$$

where $C_w = \pi(B^-wB^-)$ is the *Schubert cell* attached to w, isomorphic to $\mathbb{C}^{\ell(w)}$. We may also consider the Birkhoff decomposition

$$G = \bigsqcup_{w\in W} B^-wB,$$

which projects to the *opposite* Schubert decomposition

$$X = \bigsqcup_{w\in W} C^w, \tag{2}$$

where $C^w = \pi(B^-wB)$ is the *opposite Schubert cell* attached to w, isomorphic to $\mathbb{C}^{\ell(w_0)-\ell(w)}$. The intersection

$$\mathcal{R}_{v,w} := C^v \cap C_w \tag{3}$$

has been considered by Kazhdan and Lusztig [34] in relation with the cohomological interpretation of the Kazhdan–Lusztig polynomials. One shows [34,13] that $\mathcal{R}_{v,w}$ is non-empty if and only if $v \le w$ in the Bruhat order of W, and it is a smooth irreducible locally closed subset of C_w of dimension $\ell(w) - \ell(v)$. More recently, $\mathcal{R}_{v,w}$ has sometimes been called an open Richardson variety [36], because its closure in X is known as a Richardson variety [44].

Intersecting the decompositions (1) and (2) of X, we thus get a finer stratification

$$X = \bigsqcup_{v\le w} \mathcal{R}_{v,w}. \tag{4}$$

However, in contrast with (1) or (2), the strata $\mathcal{R}_{v,w}$ of (4) are not isomorphic to affine spaces.

1.2. Let I be the vertex set of the Dynkin diagram of G. We denote by $x_i(t)$ ($i \in I$, $t \in \mathbb{C}$) (*resp.* $y_i(t)$ ($i \in I$, $t \in \mathbb{C}$)) the one-parameter subgroups of B (*resp.* B^-) attached to the simple roots. For $K \subset I$, let B_K^- be the standard parabolic subgroup of G generated by B^- and the $x_i(t)$ with $i \in K$. We denote by $X_K = B_K^-\backslash G$ the corresponding partial flag variety. Let $\pi_K : G \to X_K$ and $\pi^K : X \to X_K$ be the natural projections, so that we have $\pi_K = \pi^K \circ \pi$. Let W_K be the parabolic subgroup of W corresponding to K with longest element w_K, and let W^K be the subset of W_K consisting of the minimal