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Integrable cluster dynamics of directed networks and pentagram maps

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A R T I C L E   I N F O

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To the memory of Andrei Zelevinsky

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A B S T R A C T

The pentagram map was introduced by R. Schwartz more than 20 years ago. In 2009, V. Ovsienko, R. Schwartz and S. Tabachnikov established Liouville complete integrability of this discrete dynamical system. In 2011, M. Glick interpreted the pentagram map as a sequence of cluster transformations associated with a special quiver. Using compatibility of Poisson and cluster structures and Poisson geometry of directed networks on surfaces, we generalize Glick's construction to include the pentagram map into a family of discrete integrable maps and we give these maps geometric interpretations. The appendix relates the simplest of these discrete maps to the Toda lattice and its tri-Hamiltonian structure.

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Contents

1. Introduction

The pentagram map was introduced by R. Schwartz more than 20 years ago [36]. The map acts on plane polygons by drawing the "short" diagonals that connect second-nearest vertices of a polygon and forming a new polygon, whose vertices are their consecutive intersection points, see Fig. 1. The pentagram map commutes with projective transformations, and therefore acts on the projective equivalence classes of polygons in the projective plane.

In fact, the pentagram map acts on a larger class of *twisted polygons*. A twisted n-gon is an infinite sequence of points $V_i \in \mathbf{RP}^2$ such that $V_{i+n} = M(V_i)$ for all $i \in \mathbf{Z}$ and a fixed projective transformation M, called the monodromy. The projective group $\mathrm{PGL}(3, \mathbf{R})$ naturally acts on twisted polygons. A polygon is closed if the monodromy is the identity.

Denote by $\mathcal{P}_n$ the moduli space of projective equivalence classes of twisted n-gons, and by $\mathcal{C}_n$ its subspace consisting of closed polygons. Then $\mathcal{P}_n$ and $\mathcal{C}_n$ are varieties of dimensions $2n$ and $2n - 8$, respectively. Denote by $T : \mathcal{P}_n \to \mathcal{P}_n$ the pentagram map (the ith vertex of the image is the intersection of diagonals (V_i, V_{i+2}) and (V_{i+1}, V_{i+3})).