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Maximal chains of isomorphic subgraphs of countable ultrahomogeneous graphs

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A B S T R A C T

For a countable ultrahomogeneous graph $\mathbb{G} = \langle G, \rho \rangle$ let $\mathbb{P}(\mathbb{G})$ denote the collection of sets $A \subset G$ such that $\langle A, \rho \cap [A]^2 \rangle \cong \mathbb{G}$. The order types of maximal chains in the poset $\langle \mathbb{P}(\mathbb{G}) \cup \{\emptyset\}, \subset \rangle$ are characterized as:
(I) the order types of compact sets of reals having the minimum non-isolated, if $\mathbb{G}$ is the Rado graph or the Henson graph $\mathbb{H}_n$, for some $n \geq 3$;
(II) the order types of compact nowhere dense sets of reals having the minimum non-isolated, if $\mathbb{G}$ is the union of μ disjoint complete graphs of size ν, where $\mu\nu = \omega$.

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1. Introduction

If $\mathbb{X}$ is a relational structure, $\mathbb{P}(\mathbb{X})$ will denote the set of domains of substructures of $\mathbb{X}$ which are isomorphic to $\mathbb{X}$. $\mathbb{X}$ is called *ultrahomogeneous* iff each isomorphism between two finite substructures of $\mathbb{X}$ can be extended to an automorphism of $\mathbb{X}$.

A structure $\mathbb{G} = \langle G, \rho \rangle$ is a *graph* iff G is a set and ρ a symmetric irreflexive binary relation on G. We will also use the following equivalent definition: a pair $\mathbb{G} = \langle G, \rho \rangle$ is a graph iff G is a set and $\rho \subset [G]^2$. Then for $H \subset G$, $\langle H, \rho \cap [H]^2 \rangle$ (or $\langle H, \rho \cap (H \times H) \rangle$, in the relational version) is the corresponding *subgraph* of $\mathbb{G}$. For a cardinal ν, $\mathbb{K}_\nu$ will denote the *complete graph* of size ν. A graph is called $\mathbb{K}_n$-free iff it has no subgraphs isomorphic to $\mathbb{K}_n$. We will use the following well-known classification of countable ultrahomogeneous graphs [9]:

Theorem 1.1 *(Lachlan and Woodrow). Each countable ultrahomogeneous graph is isomorphic to one of the following graphs*

- $\mathbb{G}_{\mu\nu}$, *the union of μ disjoint copies of $\mathbb{K}_\nu$, where $\mu\nu = \omega$,*
- $\mathbb{G}_{\text{Rado}}$, *the unique countable homogeneous universal graph, the Rado graph,*
- $\mathbb{H}_n$, *the unique countable homogeneous universal $\mathbb{K}_n$-free graph, for $n \geq 3$,*
- *the complements of these graphs.*

Properties of maximal chains in posets are widely studied order invariants (see [1,3,4,10,11]) and, as a part of investigation of the partial orders of the form $\langle \mathbb{P}(\mathbb{X}), \subset \rangle$, where $\mathbb{X}$ is a relational structure, the class of order types of maximal chains in the poset $\langle \mathbb{P}(\mathbb{G}_{\text{Rado}}), \subset \rangle$ was characterized in [7]. The aim of this paper is to complete the picture for all countable ultrahomogeneous graphs in this context and, thus, the following theorem is our main result.

Theorem 1.2. *Let $\mathbb{G}$ be a countable ultrahomogeneous graph. Then*:

(I) *If $\mathbb{G} = \mathbb{G}_{\text{Rado}}$ or $\mathbb{G} = \mathbb{H}_n$, for some $n \geq 3$, then for each linear order L the following conditions are equivalent*:
  - (a) *L is isomorphic to a maximal chain in the poset $\langle \mathbb{P}(\mathbb{G}) \cup \{\emptyset\}, \subset \rangle$;*
  - (b) *L is an $\mathbb{R}$-embeddable complete linear order with 0_L non-isolated*;
  - (c) *L is isomorphic to a compact set $K \subset \mathbb{R}$ having the minimum non-isolated.*

(II) *If $\mathbb{G} = \mathbb{G}_{\mu\nu}$, where $\mu\nu = \omega$, then for each linear order L the following conditions are equivalent*:
  - (a) *L is isomorphic to a maximal chain in the poset $\langle \mathbb{P}(\mathbb{G}) \cup \{\emptyset\}, \subset \rangle$;*
  - (b) *L is an $\mathbb{R}$-embeddable Boolean linear order with 0_L non-isolated*;
  - (c) *L is isomorphic to a compact nowhere dense set $K \subset \mathbb{R}$ having the minimum non-isolated.*