Advances in Mathematics 262 (2014) 59–114

Contents lists available at ScienceDirect

Advances in Mathematics

www.elsevier.com/locate/aim

Approximation on Nash sets with monomial singularities

Elías Baro [a,*,1], José F. Fernando [a,1], Jesús M. Ruiz [b,1]

[a] *Departamento de Álgebra, Facultad de Ciencias Matemáticas, Universidad Complutense de Madrid, 28040 Madrid, Spain*
[b] *IMI, Departamento de Geometría y Topología, Facultad de Ciencias Matemáticas, Universidad Complutense de Madrid, 28040 Madrid, Spain*

A R T I C L E I N F O

Article history:
Received 17 July 2013
Accepted 12 May 2014
Available online 28 May 2014
Communicated by Ravi Vakil

MSC:
primary 14P20
secondary 58A07, 32C05

Keywords:
Semialgebraic
Nash
Approximation
Extension
Monomial singularity
Manifold with corners

A B S T R A C T

This paper is devoted to the approximation of differentiable semialgebraic functions by Nash functions. Approximation by Nash functions is known for semialgebraic functions defined on an affine Nash manifold M, and here we extend it to functions defined on *Nash sets* $X \subset M$ *whose singularities are monomial*. To that end we discuss first *finiteness* and *weak normality* for such sets X. Namely, we prove that (i) X *is the union of finitely many open subsets, each Nash diffeomorphic to a finite union of coordinate linear varieties of an affine space*, and (ii) *every function on* X *which is Nash on every irreducible component of* X *extends to a Nash function on* M. Then we can obtain approximation for semialgebraic functions and even for certain semialgebraic maps on Nash sets with monomial singularities. As a nice consequence we show that m-dimensional affine Nash manifolds with *divisorial corners* which are class k semialgebraically diffeomorphic, for $k > m^2$, are also Nash diffeomorphic.

* Corresponding author.
E-mail addresses: eliasbaro@pdi.ucm.es (E. Baro), josefer@mat.ucm.es (J.F. Fernando), jesusr@mat.ucm.es (J.M. Ruiz).

[1] Authors supported by Spanish GAAR MTM2011-22435.

http://dx.doi.org/10.1016/j.aim.2014.05.006
0001-8708/

Contents

1. Introduction

The importance of approximation in the Nash setting arises from the fact that this category is highly rigid to work with; for instance, it does not admit partitions of unity, a usual and fruitful tool when available. On the other hand, there exist finite differentiable semialgebraic partitions of unity and therefore approximation becomes interesting as a bridge between the differentiable semialgebraic category and the Nash one. There are already relevant results concerning absolute approximation in the Nash setting (e.g. Effroymson's approximation theorem [9, §1]). An even more powerful tool is relative approximation that allows approximation having a stronger control over certain subsets. In the '80s Shiota developed a thorough study of Nash manifolds and Nash sets (see [15] for the full collected work); among other things, he devised approximation on an affine Nash manifold *relative to a Nash submanifold*. Our purpose is to generalize this type of results developing Nash approximation on an affine Nash manifold relative to a Nash subset; of course, as one can expect we need some conditions concerning the singularities on the Nash subset and we will focus on those whose singularities are of *monomial type*. This will require a careful preliminary study of this type of singularities in the Nash context. As an interesting application of our results, we prove that the Nash classification of affine m-dimensional Nash manifolds with (divisorial) corners is equivalent to the $\mathcal{C}^k$ semialgebraic classification for $k > m^2$.

Recall that a set $X \subset \mathbb{R}^n$ is *semialgebraic* if it is a Boolean combination of sets defined by polynomial equations and inequalities. A semialgebraic set $M \subset \mathbb{R}^n$ is called an (*affine*) *Nash manifold* if it is moreover a smooth submanifold of (an open subset of) $\mathbb{R}^n$; as in this paper all manifolds are affine we often drop the word affine. A function