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Redheffer type inequalities for modified Bessel functions

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Abstract. In this short note, we give new proofs of Redheffer's inequality for modified Bessel functions of first kind published by Ling Zhu (2011). In addition, using the Grosswald formula we prove new Redheffer type inequality for the modified Bessel functions of the second kind.

Keywords: Sharpening Redheffer type inequalities; Modified Bessel functions

1. INTRODUCTION

This following inequality

$$\frac{\sin x}{x} \geq \frac{\pi^2 - x^2}{\pi^2 + x^2}, \quad \text{for all } x \in \mathbb{R} \tag{1}$$

is known in literature as Redheffer's inequality [5]. J. P. Williams [7] proved the inequality (1). Chen et al. [2] obtained the following three Redheffer type inequalities for the functions $\cos x$, $\frac{\sinh x}{x}$ and $\cosh x$

$$\cos x \geq \frac{\pi^2 - 4x^2}{\pi^2 + 4x^2}, \quad x \in [0, \frac{\pi}{2}]. \tag{2}$$

$$\cosh x \leq \frac{\pi^2 + 4x^2}{\pi^2 - 4x^2}, \quad x \in [0, \frac{\pi}{2}[. \tag{3}$$

$$\frac{\sinh x}{x} \geq \frac{\pi^2 + x^2}{\pi^2 - x^2}, \quad x \in [0, \pi]. \tag{4}$$

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Recently, some extensions of inequalities (3) and (4) involving modified Bessel function have been shown in Baricz [1]. Define the function $\mathcal{I}_p : \mathbb{R} \longrightarrow [1, +\infty[$ by

$$\mathcal{I}_p(x) = 2^p \Gamma(p+1) \frac{I_p(x)}{x^p} = \sum_{n \geq 0} \frac{\left(\frac{1}{4}\right)^n}{(p+1)_n n!} x^{2n}$$

where $(p+1)_n = (p+1)(p+2)\cdots(p+n) = \frac{\Gamma(p+n+1)}{\Gamma(p+1)}$ is the well-known Pochhammer (or Appel) symbol defined in terms of Euler's gamma function, and $I_p(x)$ is the modified Bessel function. Recall that in 2007 Baricz [1] proved that for all $p > -1$, the following inequality

$$\mathcal{I}_p(x) \leq \frac{j_{p,1}^2 + x^2}{j_{p,1}^2 - x^2}, \quad x \in]0, j_{p,1}[$$

where $j_{p,n}$ is the nth positive zero of the Bessel function $J_p(x)$.

In 2008, L. Zhu and J. Sun [9] extended and sharpened inequalities (3) and (4) as follows.

Theorem 1. *Let $0 < x < r$. Then*

$$\left(\frac{r^2 + x^2}{r^2 - x^2}\right)^{\alpha} \leq \frac{\sinh x}{x} \leq \left(\frac{r^2 + x^2}{r^2 - x^2}\right)^{\beta} \tag{5}$$

holds if and only if $\alpha \leq 0$ and $\beta \geq \frac{r^2}{12}$.

Theorem 2. *Let $0 \leq x < r$. Then*

$$\left(\frac{r^2 + x^2}{r^2 - x^2}\right)^{\alpha} \leq \cosh x \leq \left(\frac{r^2 + x^2}{r^2 - x^2}\right)^{\beta} \tag{6}$$

holds if and only if $\alpha \leq 0$ and $\beta \geq \frac{r^2}{4}$.

Next, let us recall the following result which will be used in the sequel.

Lemma 1. *Let $f, g : [a, b] \longrightarrow \mathbb{R}$ two continuous functions which are differentiable on (a, b). Further, let $g' \neq 0$ on (a, b). If $\frac{f'}{g'}$ is increasing (or decreasing) on (a, b), then the functions $\frac{f(x)-f(a)}{g(x)-g(a)}$ and $\frac{f(x)-f(b)}{g(x)-g(b)}$ are also increasing (or decreasing) on (a, b).*

Proof. Denoting by $\phi(x) = \frac{f(x)-f(a)}{g(x)-g(a)}$, a simple calculation reveals that the numerator of ϕ' equals

$$\left\{ \frac{f'(x)}{g'(x)} - \frac{f(x) - f(a)}{g(x) - g(a)} \right\} g'(x)(g(x) - g(a))$$

from which the stated result follows upon applying Cauchy's mean value theorem and the monotonicity hypotheses in the lemma. ■