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A remark on compact hypersurfaces with constant mean curvature in space forms

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ABSTRACT

In this note we characterize compact hypersurfaces of dimension $n \geq 2$ with constant mean curvature H immersed in space forms of constant curvature and satisfying an optimal integral pinching condition: they are either totally umbilical or, when $n \geq 3$ and $H \neq 0$, they are locally contained in a rotational hypersurface. In dimension two, the integral pinching condition reduces to a topological assumption and we recover the classical Hopf–Chern result.

1. Introduction

The study of constant mean curvature hypersurfaces in space forms of constant curvature is one of the oldest subjects in differential geometry. There are many interesting results on this topic (see for example [13,2,14,18,16,9,8,17,20,6,3], and many others). By constructing a holomorphic quadratic differential, Hopf [13] showed that any constant mean curvature two-sphere in $\mathbb{R}^3$ is totally umbilical. Chern [8] extended Hopf's result to constant mean curvature two-spheres in three-dimensional space

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forms. Compact immersed constant mean curvature tori in $\mathbb{R}^3$ were first constructed by Wente [19].

To fix the notation, let M^n, $n \geq 2$, be a compact hypersurface with constant mean curvature H immersed in a space form $\mathbb{F}^{n+1}(c)$ of constant curvature c. Denote by h the second fundamental form of M^n and by $\mathring{h}$ its trace-free part. With this notation, M^n is totally umbilical if and only if $\mathring{h}$ vanishes. It is well known [18,9,15] that if $H = 0$ and $|\mathring{h}|^2 \leq nc$, $c > 0$, then M^n is either totally umbilical or a Clifford tori in $\mathbb{S}^{n+1}(c)$, i.e. product of spheres $\mathbb{S}^{n_1}(r_1) \times \mathbb{S}^{n_2}(r_2)$, $n_1 + n_2 = n$, of appropriate radii. This rigidity result was extended by Alencar and do Carmo [1] to hypersurfaces with constant mean curvature. The aim of this note is to show a characterization of compact hypersurfaces with constant mean curvature satisfying an *integral pinching condition* on $\mathring{h}$. This improves the result in [1]. Moreover, in dimension two, the integral inequality reduces to a topological assumption on the surface and leads to a new proof of Hopf–Chern Theorem. Our main result reads as follows:

Theorem 1.1. *Let M^n be a compact hypersurface with constant mean curvature immersed in a space form $\mathbb{F}^{n+1}(c)$ of constant curvature c. Then*

$$\int_{M^n} |\mathring{h}|^{\frac{n-2}{n}} \Big(\frac{1}{n} H^2 - |\mathring{h}|^2 - \frac{n-2}{\sqrt{n(n-1)}} |H||\mathring{h}| + nc \Big) \leq 0$$

and equality occurs if and only if M^n is either totally umbilical or, when $n \geq 3$ and $H \neq 0$, around every non-umbilical point, it is locally contained in a rotational hypersurface of $\mathbb{F}^{n+1}(c)$.

Note that, if $H = 0$ and $c \leq 0$, the statement is trivial. On the other hand, there are Clifford tori in $\mathbb{S}^{n+1}(c)$ with $|\mathring{h}|^2 \equiv nc$ that are not contained in a rotational hypersurface of $\mathbb{S}^{n+1}(c)$. Hence, the second part of the equality case in Theorem 1.1 cannot be true if $H = 0$.

In dimension two, Gauss equation and Gauss–Bonnet theorem imply that the integral inequality is equivalent to the non-positivity of the Euler characteristic of M^2 and we recover Hopf–Chern result.

Corollary 1.2 *(Hopf–Chern). Let M^2 be a compact surface with constant mean curvature immersed in a space form $\mathbb{F}^3(c)$ of constant curvature c. Then, either M^2 is totally umbilical or $\chi(M^2) \leq 0$. In particular, every compact constant mean curvature two-sphere immersed in a space form $\mathbb{F}^3(c)$ is totally umbilical.*

The proof of Theorem 1.1 relies on an improvement of the Bochner method applied to Codazzi tensors with constant trace (section 2), which was observed by the author in [7].