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Maps preserving the fixed points of triple Jordan products of operators

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Abstract

Let $\mathcal{B}(\mathcal{X})$ be the algebra of all bounded linear operators on a complex Banach space $\mathcal{X}$ with $\dim \mathcal{X} \geq 3$. In this paper, we characterize the forms of surjective maps on $\mathcal{B}(\mathcal{X})$ which preserve the fixed points of triple Jordan products of operators.

Keywords: Preserver problem; Fixed point; Triple Jordan product

1. Introduction

The study of maps on operator algebras preserving certain properties is a topic which attracts much attention of many authors. Some of these problems are concerned with preserving a certain property of sum or product of operators (see [1–8]).

Let $\mathcal{B}(\mathcal{X})$ denote the algebra of all bounded linear operators on a complex Banach space $\mathcal{X}$. For $A \in \mathcal{B}(\mathcal{X})$, denote by Lat$A$ the lattice of A, that is, the set of all invariant subspaces of A. The authors in [1] characterize the forms of maps $\phi : \mathcal{B}(\mathcal{X}) \to \mathcal{B}(\mathcal{X})$ which satisfy one of the following preserving properties:

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- $\mathrm{Lat}(A+B) = \mathrm{Lat}(\phi(A)+\phi(B))$,
- $\mathrm{Lat}(AB) = \mathrm{Lat}(\phi(A)\phi(B))$,
- $\mathrm{Lat}(AB+BA) = \mathrm{Lat}(\phi(A)\phi(B)+\phi(B)\phi(A))$,
- $\mathrm{Lat}(ABA) = \mathrm{Lat}(\phi(A)\phi(B)\phi(A))$,
- $\mathrm{Lat}(AB-BA) = \mathrm{Lat}(\phi(A)\phi(B)-\phi(B)\phi(A))$.

Recall that $x \in \mathcal{X}$ is a fixed point of an operator $A \in \mathcal{B}(\mathcal{X})$ whenever we have $Ax = x$. Denote by $F(A)$, the set of all fixed points of A. It is clear that $F(A) \in \mathrm{Lat}A$. We denote by $\dim F(T)$, the dimension of $F(T)$. Let $\dim \mathcal{X} \geq 3$. In [7], we characterized the forms of surjective maps $\phi : \mathcal{B}(\mathcal{X}) \to \mathcal{B}(\mathcal{X})$ such that $\dim F(AB) = \dim F(\phi(A)\phi(B))$, for every $A, B \in \mathcal{B}(\mathcal{X})$.

The main result of this paper is as follows.

Theorem 1.1. *Let $\mathcal{X}$ be a complex Banach space with $\dim \mathcal{X} \geq 3$. Suppose that $\phi : \mathcal{B}(\mathcal{X}) \longrightarrow \mathcal{B}(\mathcal{X})$ is a surjective map such that satisfies the following condition:*

$$F(ABA) = F(\phi(A)\phi(B)\phi(A)) \quad (A, B \in \mathcal{B}(\mathcal{X})).$$

Then $\phi(A) = \alpha A$ for every $A \in \mathcal{B}(\mathcal{X})$, where α is a complex number α with $\alpha^3 = 1$.

We recall some notations. $\mathcal{X}^*$ denotes the dual space of $\mathcal{X}$. For every nonzero $x \in \mathcal{X}$ and $f \in \mathcal{X}^*$, the symbol $x \otimes f$ stands for the rank one linear operator on $\mathcal{X}$ defined by $(x \otimes f)y = f(y)x$ for any $y \in \mathcal{X}$. Note that every rank one operator in $\mathcal{B}(\mathcal{X})$ can be written in this way. The rank one operator $x \otimes f$ is idempotent if and only if $f(x) = 1$ and is nilpotent if and only if $f(x) = 0$. We denote by $\mathcal{P}(\mathcal{X})$, $\mathcal{F}_1(\mathcal{X})$, $\mathcal{P}_1(\mathcal{X})$ and $\mathcal{N}_1(\mathcal{X})$ the set of all idempotent operators, the set of all rank one operators, the set of all rank one idempotent operators and the set of all rank one nilpotent operators in $B(X)$, respectively.

Let $x \otimes f$ be a rank one operator. It is easy to check that $x \otimes f$ is an idempotent if and only if $F(x \otimes f) = \langle x \rangle$ (the linear subspace spanned by x). If $x \otimes f$ is not idempotent, then $F(x \otimes f) = \{0\}$.

2. Maps preserving the fixed points of triple Jordan product of operators

In order to prove the main result, first we prove some auxiliary lemmas. Denote $\mathbb{C}^* = \mathbb{C} \setminus \{0, 1\}$.

Lemma 2.1. *Let $A \in \mathcal{B}(\mathcal{X})$, then $A \in \mathbb{C}^* I$ if and only if $F(PAP) = \{0\}$, for every $P \in \mathcal{P}_1(\mathcal{X})$.*

Proof. The sufficiently part is clear. If $A \notin \mathbb{C}^* I$, then there is a nonzero $x \in \mathcal{X}$ such that $x = Ax$ or x and Ax are linearly independent. Let x and Ax be linearly independent, then we can find a linear functional f such that $f(x) = f(Ax) = 1$. By setting $P = x \otimes f$, we have $F(PAP) = \langle x \rangle$ which is a contradiction. It is known that if x and Ax are linearly dependent for every $x \in \mathcal{X}$, then A is a multiple of identity.

Lemma 2.2. *Let A and B be non-scalar operators. If $F(PAP) = F(PBP)$, for every $P \in \mathcal{P}_1(\mathcal{X})$, then there exists a $\lambda \in \mathbb{C} \setminus \{1\}$ such that $B = \lambda I + (1-\lambda)A$.*

Proof. By Proposition 3.3 in [8], it is sufficient to show that $PAP \in \mathcal{P}(\mathcal{X}) \setminus \{0\}$ implies that $PBP \in \mathcal{P}(\mathcal{X}) \setminus \{0\}$ for every $P \in \mathcal{P}_1(\mathcal{X})$.

Now, let $P \in \mathcal{P}_1(\mathcal{X})$. If $PAP \in \mathcal{P}(\mathcal{X}) \setminus \{0\}$, then $F(PAP) = \mathrm{Im}\, P$ and by assumption $F(PBP) = \mathrm{Im}\, P$. Since PBP is a rank one operator, we have $PBP \in \mathcal{P}(\mathcal{X}) \setminus \{0\}$.