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A well-balanced scheme for the shallow-water equations with topography

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ABSTRACT

A non-negativity preserving and well-balanced scheme that exactly preserves all the smooth steady states of the shallow water system, including the moving ones, is proposed. In addition, the scheme must deal with vanishing water heights and transitions between wet and dry areas. A Godunov-type method is derived by using a relevant average of the source terms within the scheme, in order to enforce the required well-balance property. A second-order well-balanced MUSCL extension is also designed. Numerical experiments are carried out to check the properties of the scheme and assess the ability to exactly preserve all the steady states.

1. Introduction

During the last two decades, numerous schemes have been derived to preserve exactly (or, at least, accurately) the lake at rest. For instance, we refer the reader to the previous work by Bermudez and Vazquez [1], and next to Greenberg and LeRoux [2]. These works introduced the definition and the relevance of the well-balanced procedure. Such approaches were extended by Gosse [3] for nonlinear systems, by involving a nonlinear equation to be solved. More recently, in [4], the authors proposed a simplification (by enforcing vanishing velocities) of Gosse's work [3], yielding the so-called hydrostatic reconstruction (see [5–13] for related work).

The steady states for the shallow-water equations with nonzero discharge are known to be more difficult to exactly capture than the lake at rest configuration. The critical role played by these specific solutions was illustrated in [14], where several benchmarks were exhibited. Next, in [3], a pioneer fully well-balanced scheme was designed to deal with these sensitive steady states, where the numerical technique is based on a suitable resolution of the Bernoulli equation. Next, several methods preserving the moving steady states were designed by involving high-order accurate techniques (see [15–18] for high-order and exactly well-balanced schemes, and [19] for a high-order accurate scheme on all steady state configurations).

More recently, in [20,21], the authors have proposed an extension of the work by Gosse [3] in order to deal with Godunov-type schemes. Such Godunov-type schemes (see [22,23]) are based on approximate Riemann solvers, whose intermediate

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states are obtained by solving a Bernoulli-type equation. This process allows the authors to get a fully well-balanced scheme preserving the entropy stability. Since the resolution of the Bernoulli-type equation has a large computational cost, we adopt in the present paper a linear formulation to deal with a general form of well-balanced states. In addition, a well-balanced HLLC scheme (see [24]) is designed in [25] for turbidity currents with sediment transport and in [26] for the Ripa model. Both models are close to the shallow-water equations, and numerical methods can be easily transposed to the shallow-water case. These schemes are similar to the one presented here, since the intermediate states involve the source terms in both cases. Nevertheless, the main difference between the present work and the cited articles is the use of a HLLC-type approximate Riemann solver in [25,26], instead of the HLL-type solver we develop here. A HLLC-type solver includes more waves than a HLL-type solver, and therefore more unknowns to determine.

In the present paper, we propose a generic approach to provide a well-balanced scheme. As a consequence, the objectives of the paper are to derive a numerical scheme to approximate the solutions of the shallow-water equations, and that satisfies the following properties:

1. exact preservation of all the smooth steady states for the system with topography;
2. non-negativity preservation for the water height under the usual CFL condition;
3. ability to handle the transitions between wet and dry areas.

The paper is organized as follows. First, we devote Section 2 to the study of smooth steady states for the shallow-water equations with topography. Some comments are also given to determine steady state solutions with a dry/wet transition. Afterwards, Section 3 is dedicated to the construction of a Godunov-type scheme. We then propose a general procedure to obtain a well-balanced scheme for the shallow-water equations with a source term on the discharge equation. At this stage, the definition of the source terms is not specified, and the resulting scheme will depend on the source term discretization. In other words, the well-balance property is obtained according to the PDE governing the steady states. Next, in Section 4, we consider the particular case of the topography source term, of which we propose a suitable discretization in order to exactly preserve the steady states governed by the topography. An extension of the scheme to dry/wet transitions is also designed. A second-order extension with a MUSCL technique is then studied in Section 5, and we conclude the document with numerical experiments in Section 6. Section 7 ends the study by a brief conclusion.

2. Steady states for the shallow-water equations with topography

2.1. The shallow-water model

This paper is devoted to designing a numerical scheme to approximate the solutions of the shallow-water equations with topography. The model of interest is governed by the following system:

$$\begin{cases} \partial_t h + \partial_x q = 0, \\ \partial_t q + \partial_x \left(\dfrac{q^2}{h} + \dfrac{1}{2} g h^2 \right) = -gh\partial_x Z. \end{cases} \tag{2.1}$$

The equations (2.1) describe the behavior of water in a one-dimensional channel with a non-flat bottom. The modeled quantities are the water height $h(x,t) \geq 0$ and its depth-averaged discharge $q(x,t)$. The depth-averaged velocity u of the water is such that $q = hu$. The constant $g > 0$ stands for the gravity, while the function $x \mapsto Z(x)$ is the topography. We define the admissible states space by

$$\Omega = \left\{ W = {}^t(h,q) \in \mathbb{R}^2 \,;\, h \geq 0,\ q \in \mathbb{R} \right\}.$$

Let us note that the water height may vanish, which accounts for dry areas.

For the sake of simplicity in the notations, we rewrite (2.1) under the following condensed form:

$$\partial_t W + \partial_x F(W) = s(W, Z), \quad W \in \Omega, \tag{2.2}$$

where

$$W = \begin{pmatrix} h \\ q \end{pmatrix}, \qquad F(W) = \begin{pmatrix} q \\ \dfrac{q^2}{h} + \dfrac{1}{2} g h^2 \end{pmatrix}, \qquad s(W,Z) = \begin{pmatrix} 0 \\ -gh\partial_x Z \end{pmatrix}.$$

The homogeneous system deriving from canceling the source terms in (2.1) turns out to be hyperbolic with characteristic velocities given by $u - c$ and $u + c$ (see [27–29] for instance), where c is the sound speed, defined by

$$c = \sqrt{gh}. \tag{2.3}$$

2.2. Smooth steady states with positive water heights

In the present paper, we focus on the smooth steady state solutions of (2.2), which thus satisfy $\partial_t W = 0$. From now on, when discussing steady states, we assume a smooth topography, that is to say continuous and differentiable in space. Such