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Fully nonlinear capillary–gravity wave patterns under the tangential electric field

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ABSTRACT

Two-dimensional fully nonlinear capillary–gravity traveling waves are studied when a uniform electric field is applied in a direction parallel to the undisturbed free surface of a conducting fluid. For simplicity, we make assumption that the permittivity of the fluid is much larger than that of the up-layer gas. Therefore this two-layer problem can be reduced to a one-layer one, such that only in the fluid domain and its free boundary, the governing equations need to be solved. Fully localized solitary waves are found numerically via conformal mapping technology and direct continuation method. Unlike general capillary–gravity solitary waves on deep water that its depression branch can exit down to zero speed whilst these waves develop self-intersection points and become unphysical at low speed, under the external electric field wave-packet solitary waves can only exist at finite translating speed.

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1. Introduction

Free surface waves on conducting fluids in an external electric field arise in many physical and technological processes, including vacuum discharge, vacuum breakdown, as well as biological applications. We consider the distinct physical effects of gravity, surface tension and electrically induced forces for an inviscid, incompressible and irrotational conducting fluid. The wavelength of capillary–gravity waves in water is typically less than a few centimeters, therefore deep water assumption is physically reasonable. Normally, the external electric field is posed in the horizontal or vertical direction. Hence the dispersion relation has the following form:

$$\omega^2 = |k|(1 + k^2) + E_b k^2 \tag{1.1}$$

$$\omega^2 = |k|(1 + k^2) - E_b k^2 \tag{1.2}$$

respectively (see Section 2 for the detailed derivation). Here ω is the frequency, k represents the wave-number and $E_b > 0$ is the nondimensionalized electric strength. Linearly speaking, the horizontal external electric field tends to stabilize the surface capillary–gravity waves, while the vertical one does the opposite thing. However, since $1 + k^2 > 2|k|$, linear waves are still stable when $E_b < 2$ even if the vertical electric field is present (see for example, [1,2]). In this paper, we will consider the case when the external electric field is horizontal, and the vertical case will be studied somewhere else.

In the past several decades, two-dimensional capillary–gravity waves (with one-dimensional free surface) without electric field have been extensively studied. In the numerical aspect, the boundary integral method has been proven to

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be accurate and efficient in finding permanent wave profiles. A lot of capillary–gravity wave patterns have been computed. These results could be found, for example, in [3–6]. For the unsteady computation, a method based on time-dependent conformal mapping was derived by Dyachenko et al. (1996) in [7]. Most recently, the dynamics of capillary–gravity solitary waves are considered by Milewski et al. [8] by using the same method. A survey of the numerical methods for time-dependent irrotational water wave problems is provided by Dias and Bridges in [9].

In the case when an external electric field is present, Papageorgiou and Vanden-Broeck [10,11] studied the electrocapillary waves (the gravity effect was neglected), where the periodic traveling waves were computed via arc-length parameterization, and furthermore the dynamics of the waves were investigated by the long wave model equations. In [1,2] the capillary–gravity waves under the vertical electric field have been studied mainly by long wave approximation and for periodic waves. On the other hand, Zubarev [12,13] studied the similar problem using another approximation. He considered the gas–fluid or vacuum–fluid interface such that he can make the assumption that the permittivity of the fluid was much large compared to gas (the permittivities for water and air are 80 and 1 respectively). Therefore the actual two-layer electric field could be reduced to one layer and the exact Hamiltonian structure could be obtained and used to study the stability property of the surface waves. In this paper, we follow Zubarev's approximation, and the weakly nonlinear analysis and the fully nonlinear numerical solutions are presented.

The rest of the paper is structured as follows. In Section 2, the governing equation is described and simplified by making the assumption that the permittivity of the fluid is large, such as water. In Section 3, we consider the linear dispersion relation and derive the associated nonlinear Schrödinger equation for the quasi-monochromatic wave which will be used to predict the existence of small-amplitude, wave-packet solitary waves. In Section 4 the numerical method for computing fully nonlinear, steady surface waves in the presence of the background electric field is described, which is essentially the same to the method used in [8] but with a slight modification. Then the main numerical results are followed, including the typical profiles of solitary waves, as well as the bifurcation diagrams. It is shown that nonlinear capillary–gravity waves can only exist at finite propagating speed due to the electric field.

2. Mathematical formulation

2.1. Equations and boundary conditions

Consider the two-dimensional (2D) inviscid, incompressible and irrotational flow bounded above by a free surface and infinitely deep. The liquid is supposed to be perfectly conducting and lies in a uniform horizontal electric field E_0 which occupies the entire space. Let x be the horizontal axis, y the vertical coordinate pointing upwards and t time. We denote $y = \eta(x, t)$ as the free surface of the fluid which is reduced to $y = 0$ if there is no perturbation. We designate the velocity potential of the fluid as $\phi(x, y, t)$, then following Melcher and Schwarz [14], we can introduce the voltage potentials $\widetilde{V}(x, y, t)$ and $\widetilde{V}^+(x, y, t)$ in the fluid and in the air above the fluid respectively. The field equations are then,

$$\begin{cases} \Delta\phi = \Delta\widetilde{V} = 0, & \text{for } -\infty < y < \eta \\ \Delta\widetilde{V}^+ = 0, & \text{for } \eta < y < +\infty \end{cases} \tag{2.1}$$

where $\Delta = \partial_{xx} + \partial_{yy}$. On the interface $y = \eta(x, t)$, the voltage potential is continuous, so is its normal stress, i.e.

$$\widetilde{V} = \widetilde{V}^+ \tag{2.2}$$

$$\widetilde{\epsilon}\left(\widetilde{V}_y - \eta_x\widetilde{V}_x\right) = \epsilon^+\left(\widetilde{V}_y^+ - \eta_x\widetilde{V}_x^+\right) \tag{2.3}$$

where $\widetilde{\epsilon}$ and ϵ^+ are the electrical permittivities of the fluid and the air (or vacuum state) respectively. For the motion of the free surface, the fully nonlinear kinematic and dynamic boundary conditions are of the forms

$$\eta_t = \phi_y - \eta_x\phi_x \tag{2.4}$$

$$\rho\left(\phi_t + \frac{1}{2}\left|\nabla\phi\right|^2 + g\eta\right) + \frac{\widetilde{\epsilon}}{2}\,\frac{\eta_x^2 - 1}{\eta_x^2 + 1}\left[\left(\widetilde{V}_x\right)^2 - \left(\widetilde{V}_y\right)^2\right] - \frac{2\widetilde{\epsilon}\eta_x}{\eta_x^2 + 1}\widetilde{V}_x\widetilde{V}_y$$
$$- \frac{\epsilon^+}{2}\,\frac{\eta_x^2 - 1}{\eta_x^2 + 1}\left[\left(\widetilde{V}_x^+\right)^2 - \left(\widetilde{V}_y^+\right)^2\right] + \frac{2\epsilon^+\eta_x}{\eta_x^2 + 1}\widetilde{V}_x^+\widetilde{V}_y^+ = \sigma\,\frac{\eta_{xx}}{(1 + \eta_x^2)^{3/2}} \tag{2.5}$$

where g is the acceleration of gravity and σ is the surface tension coefficient. In order to complete the system, boundary conditions in the far field are required:

$$\phi_y \to 0, \quad \text{as } y \to -\infty \tag{2.6}$$

$$\widetilde{V}_x \to E_0, \qquad \widetilde{V}_x^+ \to E_0, \quad \text{as } |x| + |y| \to \infty. \tag{2.7}$$

Since both gravity and surface tension are considered in this paper, we can non-dimensionalize the system by choosing

$$\left[\frac{\sigma}{\rho g}\right]^{1/2}, \qquad \left[\frac{\sigma}{\rho g^3}\right]^{1/4}, \qquad \left[\frac{\sigma^3}{\rho^3 g}\right]^{1/4} \quad \text{and} \quad \left[\frac{\sigma}{\rho g}\right]^{1/2} E_0 \tag{2.8}$$