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Computation of moderate-degree fully-symmetric cubature rules on the triangle using symmetric polynomials and algebraic solving

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ABSTRACT

A novel method is presented for expressing the moment equations involved in computing fully symmetric cubature rules on the triangle, by using symmetric polynomials to represent the two kinds of invariance inherent in these rules. This method results in a system of polynomial equations that is amenable to solution using algebraic solving techniques; using Gröbner bases, rules of degree up to 15 are computed and presented, some of them new and with all their points inside the triangle.

Since all solutions to the polynomial system are computed, it is for the first time possible to prove whether a given rule type results in specific rules of a given quality; it is thus proved that for degrees up to 14 there are no non-fortuitous rules that can improve on the presented results. For degree 10, an example is also provided showing how the proposed method can be used to exclude the existence of better fortuitous rules as well.

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1. Introduction

The term "cubature" indicates the numerical computation of a multiple integral. This is an important topic in many different disciplines, with a correspondingly large body of literature. A description of the different kinds of cubature rules that exist, as well as of the mathematics used to derive them, is given in the classic book of Stroud [1], with more updated information to be found, among others, in [2] and in chapter 6 of [3]. Stroud [1] also presents a compilation of known (at the time) cubature rules, while newer rules are catalogued in [4,5] and online at the Encyclopedia of Cubature Formulas [6].

A commonly used method to derive specific cubature rules is based on moment equations and invariant theory (see [7,2] and [3, pp. 170–182]). This method, which will be used in the present paper, exploits symmetries and invariant theory to set up a non-linear system of equations, whose unknowns are the positions and weights of the integration points. The construction of the system of equations is based on Sobolev's theorem [7, see e.g.]. The use of invariants, together with appropriate algebraic computations, can lead to a significant simplification of the system of equations, which however in most cases still has to be solved numerically using an iterative method.

Although appropriate iterative numerical methods have been successfully used to obtain *individual* numerical solutions to the aforementioned system of equations, obtaining a solution in this way provides no information on its uniqueness. Conversely, inability to obtain a solution does not prove its inexistence (though it is a strong indication, when sufficiently robust numerical methods are employed). It is thus interesting and useful to be able to perform an exhaustive computation that provides all the solutions for a cubature rule.

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In this paper we focus on fully symmetric cubature rules on the triangle, for which many specific rules have already been presented in the literature [1,8–15]. Extending significantly the results given in analytic form by Lyness and Jespersen [9], we provide results for cubature rules of degree up to 15. Symmetric polynomials [16] are used in generating the moment equations, to represent the two kinds of invariance inherent in these rules. This leads to a system of equations which is amenable to algebraic solving, thus allowing all cubature rules of a given type to be computed.

In Section 2 we present concisely the concepts of symmetric polynomials, areal coordinates and polynomial system solving that will be used in the rest of the paper. Section 3 presents the derivation of the moment equations for fully symmetric rules. While the overall derivation follows the one by Lyness and Jespersen [9], symmetric polynomials are used here to express the invariance with respect to permutation of points within an orbit, resulting in expressions that are better suited to algebraic manipulation than those previously reported in the literature.

Instead of using an iterative solver to find a numerical approximation of a single solution of the moment equations, as usually done in the literature, in Section 4 we further transform the moment equations to take into account their invariance with respect to permutations of orbits of the same type (once more, using symmetric polynomials to express the invariance). This invariance (which to the author's knowledge has not been exploited before in the relevant literature) is key in providing a new form of the moment equations that, though not explicitly given as the previous one, is actually amenable to algebraic solving.

Section 5 summarises the cubature rules thus obtained using algebraic solving techniques and comments on the main features of the provided results, among which there are new rules which match (though they do not exceed) existing ones in terms of quality and number of points. Algebraic solving allows (for the first time in the non-trivial cases) the computation of all cubature rules of a given type, thus another important result obtained here is the non-existence of non-fortuitous cubature rules that improve on the ones presented in terms of quality and number of points. The case of fortuitous rules is also considered. Finally, Section 6 concludes by pointing out the main results obtained in the paper.

2. Theoretical background

2.1. Symmetric polynomials

The formulation presented in this paper is based on invariant theory and in particular it uses the theory of symmetric polynomials [16]. As we will see in the following, the use of symmetric polynomials provides an initial concise formulation of the non-linear system of equations, while also leading to simpler computation and presentation of the solution.

A *symmetric polynomial* is a multivariate polynomial in n variables, say $x_1, x_2, \dots, x_n$, which is invariant under any permutation of its variables. For example, the polynomial $x_1x_2 + x_2x_3 + x_3x_1$ is a symmetric polynomial of degree 2 in the three variables x_1, x_2 and x_3, as can be easily seen by swapping any two variables.

We define the *elementary symmetric polynomials* $\tilde{x}_k$ as the sums of all products of k distinct variables x_i, with negative sign when k is odd, that is

$$\tilde{x}_k = (-1)^k \sum_{i_1<i_2<\cdots<i_k} x_{i_1}x_{i_2}\cdots x_{i_k} \tag{1}$$

with $\tilde{x}_0 = 1$. The alternating sign $(-1)^k$ in Eq. (1), which does not appear in the usual definition of the elementary symmetric polynomials, is introduced here as it leads to simpler expressions. While elementary symmetric polynomials are usually denoted using a letter (e.g. Π_k, s_k or e_k) which is different from the variable name, we use here the superimposed tilde over the variable name since we will be dealing with elementary symmetric polynomials of different sets of variables.

The *fundamental theorem of symmetric polynomials* states that any symmetric polynomial in the variables x_i can be uniquely expressed as a polynomial in the elementary symmetric polynomials $\tilde{x}_k$ [17, p. 118]. This obviously holds true independently of the presence of the alternating sign in Eq. (1). The proof of the fundamental theorem also provides an algorithm for *symmetric reduction*, that is for expressing arbitrary symmetric polynomials in terms of the elementary symmetric polynomials.

Eq. (1) allows computing the elementary symmetric polynomials $\tilde{x}_k$ in terms of the n variables x_i. Conversely, the values x_i can be calculated [17, p. 89] from $\tilde{x}_k$ as the solutions for x of the polynomial equation

$$\sum_{j=0}^{n} \tilde{x}_{n-j}x^j = 0 \tag{2}$$

2.2. Areal coordinates

Consider the generic triangle shown in Fig. 1, defined through its three vertices with Cartesian coordinates (x_1, y_1), (x_2, y_2) and (x_3, y_3). For a point P with Cartesian coordinates x and y, we define the areal coordinates L_1, L_2 and L_3 (see

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