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A second order in time local projection stabilized Lagrange–Galerkin method for Navier–Stokes equations at high Reynolds numbers

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ABSTRACT

We present a stabilized Backward Difference Formula of order 2-Lagrange Galerkin method to integrate the incompressible Navier–Stokes equations at high Reynolds numbers (Re). The stabilization of the conventional Lagrange–Galerkin method is done via a local projection technique for inf–sup stable finite elements. We prove that for a finite time T the a priori error estimate for velocity in a mesh dependent norm is $O(h^m + \Delta t^2)$, whereas the error for pressure in the $l^2(L^2(D))$ norm is $O(h^m + \Delta t^2)$, with error constants that are independent of the Re^{-1}; here, m denotes the degree of the polynomials of the velocity finite element space. The size of the stabilization parameters is calculated from the velocity error estimate in a way that the error is optimal when the solution is sufficiently smooth. Numerical examples at high Reynolds numbers show the robustness of our method.

1. Introduction

In this paper, we present a local projection stabilized Lagrange–Galerkin (LG) method to integrate the time dependent Navier–Stokes (NS) equations at high Reynolds numbers (Re). LG methods were introduced in [1,2] and combine a discrete Galerkin projection method, usually finite elements, with a backward discretization of the material derivative along flow trajectories. The application of these methods to integrate NS equations has some advantages such as numerical stability and the way of dealing with the nonlinear terms $u \cdot \nabla u$. It is known that in the integration of NS equations by conventional implicit time marching schemes, the nonlinear terms yield an algebraic system of nonlinear equations the solution of which is achieved by an iterative method; thus, this procedure may become expensive in terms of computer memory and CPU time. In contrast, backward integration of the material derivative along trajectories is a natural way of introducing upwinding in the space discretization of the equations, and transforms the NS equations into a linear Stokes problem, so at each time step one has to solve an algebraic linear system of equations that is more manageable than the algebraic nonlinear system of equations produced by conventional implicit time marching schemes; moreover, we must remark that upwinding along the trajectories is a numerical mechanism to stabilize the convective terms. A priori, these advantages make LG methods look like efficient methods, but they have a drawback concerned with the calculation of some integrals which appear in their formulation. In general, such integrals cannot be evaluated exactly because their integrands are the product of functions

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defined in two different meshes, so quadrature rules must be used; and such rules have to be of high order because the stability and overall accuracy of LG methods require the above mentioned integrals to be calculated very accurately, see [3,4]. Since each quadrature point has an associated foot of characteristic that is calculated by solving backward in time a system of ordinary differential equations, then at each time step many systems have to be solved by a numerical method incorporating a point searching algorithm to identify the element of the mesh where each foot is located; the location of points inside the elements is a trivial task when the spatial mesh is composed of squares or uniform hexahedra, but if the mesh is unstructured the location of points is not that simple; hence, LG methods may become less efficient than they look at first. Recently, [4,5] have formulated a modified LG method intended to partially fix such a drawback. Another procedure to make more efficient and accurate LG methods is their stabilization via a local projection stabilization (LPS) technique.

It is relatively easy to prove that LG methods are unconditionally stable if the aforementioned integrals are calculated exactly; however, it is observed that when the time step, Δt, is small, that is, at small CFL numbers, and the viscosity is not sufficiently high there are disjoint intervals of values of Δt in which the solution becomes either unstable or significantly less accurate; of course, this strange behavior at low CFL numbers disappears if the integrals are calculated exactly or with high accuracy. For convection-dominated diffusion equations it is shown in [6] that if LG methods are stabilized via the subgrid viscosity method of [7] the stabilized LG method solutions are more accurate than the solutions of conventional LG methods. Based on this experience for convection–diffusion equations, we propose in this paper the stabilization of LG methods in the spirit of the LPS approach of [8–10], just to cite a few. This stabilization technique is well suited to LG methods for the following reasons: (1) it does not break up the Stokes problem structure of conventional LG methods, for LPS is a symmetric stabilization technique acting only on parts of the residual, in contrast with the stabilization by SUPG method that uses the full residual; (2) it is flexible in the sense that can be used with any conventional time marching scheme; (3) it is relatively easy to incorporate in any conventional LG method code; (4) as mentioned in [10,8], the LPS technique is related to the subgrid viscosity method, but from a computational point of view we found LPS to be easier to implement for our purpose.

In this paper, the local projection stabilization is applied with inf–sup stable conforming finite elements that support the definition of the quasi-local interpolation operator introduced in [11], this operator preserves the discrete divergence and has the same approximation properties as a standard interpolation operator. The pairs of conforming finite elements for which such an interpolation operator exists are $(P_m,\ P_{m-1})$, $m \geq 2$ in two dimensions, and $m \geq 3$ in three dimensions, and the generalized mini-element for $m \geq 1$ in dimension two or three, see [11]. We calculate, under mild regularity assumptions, error estimates for the velocity and pressure that do not depend on Re^{-1}.

The paper is organized as follows. In Section 2, we present the continuous problem, its weak formulation, and its backward in time discretization, along flow trajectories, by means of the Backward Difference Formula of order 2 (BDF2). We introduce the formulation of the local projection stabilized LG-BDF2 method in Section 3. The analysis on the stability of the new method in the mesh dependent norm associated with the local projection stabilized bilinear form is performed in Section 4. Section 5 is devoted to the error analysis. Finally, in order to test the ability of the method in flows at high Reynolds numbers, we run some numerical experiments in Section 6.

We introduce some notation about the functional spaces we use in the paper. For $s \geq 0$ real and real $1 \leq p \leq \infty$, $W^{s,p}(D)$ denotes the real Sobolev spaces defined on D for scalar real-valued functions. $\|\cdot\|_{W^{s,p}(D)}$ and $|\cdot|_{W^{s,p}(D)}$ denote the norm and semi-norm, respectively, of $W^{s,p}(D)$. When $s = 0$, $W^{0,p}(D) := L^p(D)$. For $p = 2$, the spaces $W^{s,2}(D)$ are denoted by $H^s(D)$, which are real Hilbert spaces with inner product $(\cdot,\cdot)_s$. For $s = 0$, $H^0(D) := L^2(D)$, the inner product in $L^2(D)$ is denoted by $(\cdot,\cdot)$. $H_0^1(D)$ is the space of functions of $H^1(D)$ which vanish on the boundary ∂D in the sense of trace. H^{-1} denotes the dual of $H_0^1(D)$. The corresponding spaces of real vector- and tensor-valued functions, $v : D \to \mathbb{R}^d$, $d > 1$ integer, are denoted by boldface letters; for instance, $\mathbf{W}^{s,p}(D) := (W^{s,p}(D))^d := \{v : D \to \mathbb{R}^d : v_i \in W^{s,p}(D),\ 1 \leq i \leq d\}$. Let X be a real Banach space $(X, \|\cdot\|_X)$, if $v : (0,T) \to X$ is a strongly measurable function with values in X, we set $\|v\|_{L^p(0,t;X)} = \left(\int_0^t \|v(\tau)\|_X^p \, d\tau\right)^{1/p}$ for $1 \leq p < \infty$, and $\|v\|_{L^\infty(0,t;X)} = \operatorname{ess\,sup}_{0<\tau\leq t} \|v(\tau)\|_X$; when $t = T$, we shall write, unless otherwise stated, $\|v\|_{L^p(X)}$. We shall also use the following discrete norms:

$$\|v\|_{l^p(X)} = \left(\Delta t \sum_{i=1}^{N} \|v(\tau_i)\|_X^p\right)^{1/p}, \qquad \|v\|_{l^\infty(X)} = \max_{1\leq i\leq N} \|v(\tau_i)\|_X \,,$$

where $l^p(X)$ is a shorthand notation for the space $l^p(0,T;X)$, $1 \leq p \leq \infty$, defined as

$$l^p(0,T;X) := \left\{v : (0, t_1, t_2, \ldots, t_N = T) \to X : \ \|v\|_{l^p(X)} < \infty\right\}.$$

Finally, we shall also make use of the space of continuous and bounded functions in time with values in X denoted by $C([0,T];X)$, and the space $C^{r,1}(\overline{D})$, $r \geq 0$, of functions defined in the closure of D, r times differentiable and with the rth derivative being Lipschitz continuous.

Throughout this paper, C will denote a generic positive constant which is independent of both the space and time discretization parameters h and Δt respectively. C will have different values at different places of appearance. Also, in many places the expression elementary inequality denotes the Cauchy's inequality $ab \leq \frac{\epsilon}{2}a^2 + \frac{1}{2\epsilon}b^2$ $(a,\ b > 0,\ \epsilon > 0)$.