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Convergence of equilibria for numerical approximations of a suspension model

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ABSTRACT

In this paper we study the numerical approximations of a non-Newtonian model for concentrated suspensions.

First, we prove that the approximative models possess a unique fixed point and study their convergence to a stationary point of the original equation.

Second, we implement an implicit Euler scheme, proving the convergence of these approximations as well.

Finally, numerical simulations are provided.

1. Introduction

Non-Newtonian (or complex) fluids often appear in nature and industry. Good examples of such fluids are toothpaste, ketchup, magma, blood, mucus or emulsions such as mayonnaise among many others. A special type of complex fluids are concentrated suspensions, which can be found, for example, in medicine (blood) or in building industry (cement). The dynamical behaviour of suspensions is still far from being well understood as developing a faithful mathematical model of such processes is not an easy task.

We are interested in an equation modelling suspensions which was proposed in [1]. In the last years, several authors have studied for this equation the existence and uniqueness of solutions [2,3], the asymptotic behaviour [4,5] and numerical approximations [6–8].

In our previous paper [8] we studied a sequence of approximative problems for this model, in which finite-difference schemes were used to deal with the partial derivative with respect to the spatial variable. The problem was split in three steps: a partial differential equation with a large diffusion, an infinite system of ordinary differential equations and finally a finite system of ordinary differential equations. For initial data satisfying suitable assumptions it was proved that the iterate limit of the solutions of the approximative problems in the space $C([0,T], L^2(\mathbb{R}))$ is equal to the solution of the original equation.

In this paper we extend the results from [8] in two ways.

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First, we study the convergence of the fixed points of the approximative problems. It is well-known [2] that for certain values of the parameters of the equation there exists a unique fixed point of the problem with support not included in the interval $[-1, 1]$. This equilibrium is asymptotically stable [4] and the numerical simulations in [7] suggest that every solution with initial data with support not included in $[-1, 1]$ converges to this fixed point as time goes to $+\infty$. We prove that each of the approximative problems possesses a unique fixed point and also that the iterate limit of the equilibria of the approximative problems in the space $L^2(\mathbb{R})$ is equal to the equilibrium of the original equation with support not included in the interval $[-1, 1]$.

Second, we complete the sequence of approximations of the problem by implementing an implicit Euler scheme for the discretization of the time derivative. We prove that the solution of the resulting system converges in the space $C([0, T], L^2(\mathbb{R}))$ to the solution of the finite system of ordinary differential equations approximating the original equation.

Finally, some numerical simulations are provided in the last section.

2. Previous results

In the previous paper [8] the authors considered the convergence of finite-difference approximations of the problem

$$\frac{\partial p}{\partial t} - D(p(t))\frac{\partial^2 p}{\partial \sigma^2} + \frac{1}{T_0}\chi_{\mathbb{R}\setminus[-1,1]}(\sigma)\, p = \frac{D(p(t))}{\alpha}\delta_0(\sigma), \tag{1}$$

$$p \geq 0, \quad p(0, \sigma) = p^0(\sigma), \tag{2}$$

where $p = p(t, \sigma)$, $t \in [0, T]$, $\sigma \in \mathbb{R}$, T_0 and α are positive constants.

Here, δ_0 is the Dirac δ-function with support in the origin,

$$D(p(t)) = \frac{\alpha}{T_0}\int_{|\sigma|>1} p(t, \sigma)\, d\sigma$$

and χ_I is the indicator function in the interval I.

The function $p(t, \sigma)$ is a probability density at time t, so for any $t \in [0, T]$,

$$\int_{\mathbb{R}} p(t, \sigma) d\sigma = 1, \qquad p(t, \sigma) \geq 0, \quad \text{for a.a. } \sigma \in \mathbb{R}. \tag{3}$$

It is well-known [2] that for any $p^0 \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$ such that $p^0 \geq 0$ a.e., $\int_{\mathbb{R}} p^0(\sigma) d\sigma = 1$, $\int_{\mathbb{R}} |\sigma|\, p^0(\sigma) d\sigma < \infty$ and $D(p_0) > 0$ there exists a unique solution $p = p(t, \sigma)$ of problem (1)–(2), which satisfies (3).

We consider as a first step the approximative problem

$$\partial_t p^c - \left(D\left(p^c(t)\right) + \frac{1}{c}\right)\partial^2_{\sigma\sigma} p + \frac{1}{T_0}\chi_{\mathbb{R}\setminus[-1,1]}(\sigma)\, p^c = \frac{D(p^c(t))}{\alpha}\delta_c(\sigma), \tag{4}$$

$$p^c \geq 0, \quad p^c(0, \sigma) = p^0_c(\sigma), \tag{5}$$

where $p^c = p^c(t, \sigma)$, $c > 0$ is a large parameter and the δ-function δ_0 is replaced by the step continuous from the right function

$$\delta_c(\sigma) = \begin{cases} 0, & \text{if } \sigma < -\dfrac{1}{2c}, \\ c, & \text{if } -\dfrac{1}{2c} \leq \sigma < \dfrac{1}{2c}, \\ 0, & \text{if } \sigma \geq \dfrac{1}{2c}. \end{cases}$$

We would like to highlight the fact that the new term $\frac{1}{c}\partial^2_{\sigma\sigma} p$ is an artificial diffusion which helps us to prove the convergence of the approximative solutions. Such a trick is very common in the numerical approximations of problems in Physics. Also, $[-\frac{1}{2c}, \frac{1}{2c}]$ is the support of the map δ_c, which approximates the δ-function δ_0. Therefore, when $c \to +\infty$, the artificial diffusion and the support of δ_c converge to 0 in unison.

Let p^0_c be such that

$$p^0_c \in C^\infty_0(\mathbb{R}), \quad p^0_c \geq 0 \text{ a.e.}, \int_{\mathbb{R}} p^0_c(\sigma) d\sigma = 1, \tag{6}$$

$$p^0_c \to p^0 \quad \text{in } L^2(\mathbb{R}), \qquad \sigma p^0_c \to \sigma p^0 \quad \text{in } L^1(\mathbb{R}), \text{ as } c \to +\infty. \tag{7}$$