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Proofs of the stability and convergence of a weakened weak method using PIM shape functions

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ABSTRACT

Recently, the smoothed point interpolation method (S-PIM) regarded as a weakened weak (W^2) formulation method has been developed for solving engineering mechanics problems. It works well with distorted meshes. The *G* space theory offers the theoretical base for all the W^2 methods that use smoothing operations. In this paper, we first prove mathematically that if a function is of Lipschitz continuity and its interpolated function is established using PIM shape functions, then the interpolated function belongs to a $G^s_{h,0}$ space. Our proofs work for smoothing operations that are the node-based, cell-based and a mixture of both smoothing domains. In addition, when mesh is refined under a given regularity condition, a sufficiently smooth target function can be approximated by its interpolated function with arbitrary accuracy, meaning that the interpolation error norm approaches to zero. Therefore, the stability and convergence of a W^2 method using PIM shape functions and *G* space theory can be ensured.

1. Introduction

The finite element method (FEM) [1] developed in the past six decades for numerical simulations of various engineering systems is an important and practical numerical approach using the weak form. It has been applied to heat transfer, mechanics of materials, electromagnetics, fluid dynamics, acoustics systems and so on. It can simulate efficiently these phenomena for reliable solutions, and assist engineering designs. However, there exist some drawbacks in the FEM, such as the need for quality mesh that is costly to generate, and significant loss of accuracy caused by distortions of the mesh. It does not work well with linear triangular (T3) elements which are easily created.

In order to solve these problems, a variety of meshfree methods have been developed, such as Kansa method [2], the method of fundamental solutions (MFS) [3,4], the singular boundary method (SBM) [5,6], which far more rich than the FEM. It can achieve high accuracy solution, but often at the cost of the efficiency of computation. Recently, numerical methods based on the W^2 formulation [7] of partial differential equations (PDEs) have been proposed. The smoothed finite element (S-FEM) [8–10] is a typical set of methods developed using smoothing domains (SDs). The SDs are generated on the top of a FEM mesh. Based on types of SDs, various S-FEMs, including CS-FEM [10,11], NS-FEM [12–16], ES-FEM [17–21] and FS-FEM, have been established [8].

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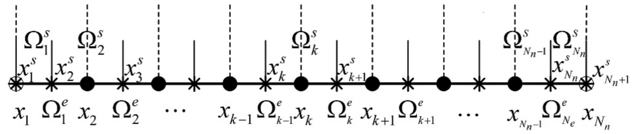

Fig. 1. Division of 1D problem domain Ω: the background mesh with N_e elements and N_n nodes (•); Node-based SDs with $N_s = N_n$ SDs and $N_n + 1$ points (∗) which are placed on the centers of elements.

Using the W^2 formulation, the consistency requirement and the differential order of the assumed field functions are reduced. The G^s space defined in the Ref. [7] contains functions that may be non-differentiable, even discontinuous, and can be used to formulate the S-PIM [22]. Various S-PIM models, including CS-PIM [23], NS-PIM [24–26], ES-PIM [27–29], and FS-PIM, have been developed on the basis of SDs. Each of these S-PIM models possesses its unique features or properties. The G space theory [7] offers a theoretical base to these S-PIM models. However, proofs of the stability and convergence of these methods using the G space theory are not given mathematically. Therefore, the further study of the theory and properties of the G space is of great importance.

In this work, a rigorous mathematical study on a $G^s_{h,0}$ space constructed using PIM shape functions is conducted. We first prove that a function interpolated using continuous PIM shape functions based on elements belongs to a $G^s_{h,0}$ space for a given target function of Lipschitz continuity. Especially, the G^s semi-norm of the interpolated function is bounded, as mesh is refined to the extreme as nested fashion. It follows that for a given target function of Lipschitz continuity, a function interpolated using discontinuous PIM shape functions based only on nodes (in meshfree settings) still belongs to a $G^s_{h,0}$ space, but it is discontinuous. Our proofs are on the node-based, cell-based and a mixture of both smoothing domains which provides the stability of numerical solutions from all the W^2 methods using PIM shape functions. We then prove that an interpolated function using continuous/discontinuous PIM shape functions can approximate a sufficient smooth target function in G^s norm, which provides the convergence of numerical solutions from all the W^2 methods using PIM shape functions.

2. $G^s_{h,0}$ spaces

G space theory is a foundation of the W^2 formulation using smoothing operation [30]. It is of importance for all the W^2 methods with both meshfree and FEM settings, including S-PIM, S-FEM. In the G space theory, a subspace $G^s_{h,0}$ of G^s space is a discrete space, which is defined in later section.

2.1. Smoothing domains and shape functions

In a discrete numerical method, such as FEM, a n-dimensional problem domain $\Omega \in R^n$ is divided into N_e non-overlapping and no-gap elements $\Omega^e_1, \Omega^e_2, \ldots, \Omega^e_{N_e}$. On this element mesh, the domain can be further divided into N_s non-overlapping and no-gap SDs [8]. Based on the Ref. [8], we have the following remark:

Remark 1. The sufficient conditions of establishing SDs:

(1) All SDs must be non-overlapping and no-gap.
(2) All SDs do not cover beyond two nodes.
(3) The number of SDs is larger than or equal to its minimum value $N_s^{\min}$ [8].
(4) Dimensions of SDs must satisfy a given regularity condition.

For example, 1D problem domain following Remark 1 can be divided into $N_s = N_n$ SDs and $N_n + 1$ smoothing points/boundaries, which is called node-based SDs and is shown in Fig. 1. Except the node-based SDs, other types of SDs are given in Section 3.4.

For high dimension problem domain, different types of SDs have been also established (see details Ref. [8]), such as, face-based, edge-based node-based and cell-based SDs.

Consider a solid mechanics problem over domain Ω. If a smoothing domain is established following Remark 1, the target function can be approximated by an interpolated function $w_I(x)$ as follows

$$w_I(x) = \sum_{i=1}^{N_n} \phi_i d_i, \tag{1}$$

where d_i and ϕ_i, respectively, denote the nodal value and nodal shape function of the target function at the node located at x_i. Therefore, the interpolated function at nodes satisfies

$$w_I(x_i) = w(x_i) = d_i, \quad i = 1, 2, \ldots, N_n,$$

where $x_i,\ i = 1, 2, \ldots, N_n$ are nodes of a background mesh (that is a usual FEM mesh).

In this paper, the following two kinds of one dimensional nodal shape functions are employed.