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Numerical solutions of elliptic partial differential equations using Chebyshev polynomials

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A B S T R A C T

We present a simple and effective Chebyshev polynomial scheme (CPS) combined with the method of fundamental solutions (MFS) and the equilibrated collocation Trefftz method for the numerical solutions of inhomogeneous elliptic partial differential equations (PDEs). In this paper, CPS is applied in a two-step approach. First, Chebyshev polynomials are used to approximate a particular solution of a PDE. Chebyshev nodes which are the roots of Chebyshev polynomials are used in the polynomial interpolation due to its spectral convergence. Then the resulting homogeneous equation is solved by boundary type methods including the MFS and the equilibrated collocation Trefftz method. Numerical results for problems on various irregular domains show that our proposed scheme is highly accurate and efficient.

1. Introduction

Over the past couple of decades, meshless methods have been gaining popularity in engineering and scientific computing. Many researchers in mathematics and engineering have successfully applied meshless methods to solve many challenging problems in their fields [1–4]. In recent years meshless methods have become a useful alternative to traditional methods such as finite element method (FEM) and finite difference method (FDM). There are two types of meshless methods: domain-type and boundary-type. The domain type methods are evolved from the FEM whereas the boundary-type methods are from the boundary element method (BEM). One of the common goals of developing meshless methods is to solve a given set of partial differential equations (PDEs) with minimum human and computational costs. Hence, other than the accuracy, the simplicity and efficiency of a meshless algorithm is also of great importance.

Let us consider the following boundary value problem:

$$\mathcal{L}u(x,y) = f(x,y), \quad (x,y) \in \Omega, \tag{1}$$

with Dirichlet boundary condition

$$u(x,y) = g(x,y), \quad (x,y) \in \partial\Omega_1, \tag{2}$$

and Neumann boundary condition

$$\frac{\partial u(x,y)}{\partial \mathbf{n}} = h(x,y), \quad (x,y) \in \partial\Omega_2, \tag{3}$$

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where $\Omega \subset R^2$ is a simply connected domain bounded by a simple closed curve, $\partial\Omega_1$ and $\partial\Omega_2$ are parts of the boundary $\partial\Omega$ with $\partial\Omega_1 \cup \partial\Omega_2 = \partial\Omega$ and $\partial\Omega_1 \cap \partial\Omega_2 = \varnothing$, $\mathcal{L}$ is an elliptic differential operator, f, g and h are given functions, and the function f can be smoothly extended to a rectangular domain containing Ω.

In this paper, we use an approach that combines a particular solution method and a boundary method to solve the problem (1)–(3). Let $u = u_p + u_h$ be the solution of (1)–(3) where u_p is a particular solution that satisfies (1), and its associated homogeneous solution $u_h(x, y)$ satisfies,

$$\mathcal{L}u_h(x, y) = 0, \qquad (x, y) \in \Omega, \tag{4}$$

$$u_h(x, y) = g(x, y) - u_p(x, y), \qquad (x, y) \in \partial\Omega_1, \tag{5}$$

$$\frac{\partial u_h(x, y)}{\partial \mathbf{n}} = h(x, y) - \frac{\partial u_p(x, y)}{\partial \mathbf{n}}, \qquad (x, y) \in \partial\Omega_2. \tag{6}$$

The basic idea is to transform the given inhomogeneous problem into a homogeneous one via a numerically obtained particular solution to the inhomogeneous equation. A particular solution satisfies the given differential equation (1) in the infinite domain, but it does not necessarily satisfy the given boundary conditions (2)–(3).

In 1967, Fox et al. [5] proposed the method of particular solutions (MPS) for solving elliptic eigenvalue problems using the combination of Bessel functions and sine functions as basis functions. In the recent decades, many different numerical techniques of the method of particular solutions have been developed for solving partial differential equations [1,2,6–12]. In these approaches, a variety of basis functions have been used. Among them, radial basis functions (RBFs) have been very popular due to their simplicity and effectiveness in implementation. Typically, there are two approaches for constructing the particular solution $\Phi(r)$ that satisfies the equation

$$\mathcal{L}\Phi(r) = \phi(r). \tag{7}$$

One approach is to utilize the RBFs as the basis functions for constructing the particular solution $\Phi(r)$ through collocation method. The other approach is to use the RBFs as the basis functions for approximating $\phi(r)$ and for deriving a particular solution $\Phi(r)$ from Eq. (7) by reverse differentiation process. We remark that the first approach using RBFs does not guarantee the matrix invertibility [13] while the matrix by the second approach possesses the positive definite property [14]. Kansa's method [2] as a popular method in the meshless literature belongs in the first approach, while the method of approximate particular solutions (MAPS) in [1,9,12] belong in the second approach. In this paper, we employ the first approach to obtain a particular solution, but we use Chebyshev polynomials as basis functions.

There exist different methods of using polynomials [8,11,15–17] for the evaluation of approximate particular solutions. Chen et al. in [8] obtained particular solutions in analytical form for 2-D Poisson equation when the forcing term f is a homogeneous polynomial. Golberg et al. in [11] implemented MPS using Chebyshev interpolants for 2-D and 3-D Helmholtz type equations when the forcing term is monomial. However, book keeping of the many monomial terms of the approximate polynomial $\widehat{f}$ and the particular solutions corresponding to these terms becomes very tedious and inefficient. Chen et al. [15] used a finite geometric series expansion on a differential operator to directly obtain a particular solution corresponding to each Chebyshev polynomials. However, the procedure of the actual implementation is quite cumbersome since some tedious algebraic operations require symbolic manipulations that impede the computational efficiency. Karageorghis and Kyza [17] derived particular solutions for Poisson and bi-harmonic equations directly using Chebyshev polynomials as basis functions. In [17], they made use of special properties of the second derivatives of Chebyshev polynomials without the need for the expanded form of the polynomials. They used approach two for the MPS, and in the solution process they needed to solve a block matrix system, the dimension of which is big when the degree of the Chebyshev polynomials used is high. Ding et al. in [16] proposed a recursive formulation or a matrix free method to derive a particular solution. This method requires the expansion of each of the Chebyshev polynomials which comprises the approximate forcing term. To avoid the expansion of the Chebyshev polynomials, we propose a particular solution method that combines the direct collocation and the reduction of the second order derivative of Chebyshev polynomials. The method is easy in coding and implementation. We use the Gauss–Lobatto nodes to ensure the spectral convergence for the Chebyshev interpolation. Also there is no problem of ill-conditioning which occurs when RBF approximations are used.

Once a particular solution is found, we continue to solve the associated homogeneous problem (4)–(6) by boundary methods. We use one of the two options: the method of fundamental solution and the equilibrated approach in collocation Trefftz method (CTM). The MFS was first proposed in 1964 by Kupradze and Aleksidze [18] and it has been used for solving various problems since then. The high accuracy of both MFS and the MPS of Chebyshev polynomials helps produce a highly accurate solution of (1)–(3). Due to the fact that MFS requires a fictitious boundary in the solution process, we use the CTM [19] as an alternative. The CTM is based on T-complete functions and does not require a fictitious boundary for solving the homogeneous problem. To treat the highly ill-conditioned linear system resulting from the direct implementation of CTM, we have adopted the equilibrated matrix concept [3,20] to determine the scales and to construct an equivalent linear algebraic problem with a leading matrix less ill-conditioned.

The organization of this paper is as follows: In Section 2, we briefly discuss the properties of the Chebyshev polynomials. Then we use the Chebyshev collocation method to find an approximate particular solution with the change of variables and re-scaling of PDEs. In Section 3, we describe the standard boundary-type meshless methods MFS and CTM, and the incorporation of equilibrated matrix approach in CTM. The numerical results for verification of the accuracy of our proposed method is presented in Section 4. Conclusions drawn from the work and the numerical results are given in Section 5.