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Inverse-free recursive multiresolution algorithms for a data approximation problem

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ABSTRACT

We present inverse-free recursive multiresolution algorithms for data approximation problems based on energy functionals minimization. During the multiresolution process a linear system needs to be solved at each different resolution level, which can be solved with direct or iterative methods. Numerical results are reported, using the sparse Cholesky factorization, for two applications: one concerning the localization of regions in which the energy of a given surface is mostly concentrated, and another one regarding noise reduction of a given dataset. In addition, for large-scale data approximation problems that require a very fine resolution, we discuss the use of the Preconditioned Conjugate Gradient (PCG) iterative method coupled with a specialized monolithic preconditioner, for which one preconditioner is built for the highest resolution level and then the corresponding blocks of that preconditioner are used as preconditioners for the forthcoming lower levels.

1. Introduction

Several multilevel techniques have been recently proposed, for solving data approximation problems, which include the use of normalized hierarchical basis functions, energy functional minimization, wavelet based regularization, and domain decomposition; see, e.g. [1–5]. In particular, in [2] and references therein a method to obtain a $\mathcal{C}^1$-spline surface approximating a given dataset by minimizing a certain "energy functional" is considered. It is shown that, under certain conditions, such approximating surface exists and is unique. A convergence result is also established and some numerical and graphical examples are given. Later, in [3], a multiresolution scheme associated to the problem stated in [2] is developed. More precisely, the data approximation problem is considered for different resolution levels which are defined through a sequence of nested triangulations of the domain over which the surface is defined. The main contribution of [3] is the development of multiresolution algorithms (decomposition and reconstruction) in such a way that the minimal energy approximating splines obtained at different levels may be related. As an application, two examples regarding the noise reduction and the localization of regions in which the energy of a given surface is mostly concentrated are considered. Nevertheless, the algorithms developed in [3] need to compute the inverse of large and very ill-conditioned matrices, impeding to carry the considered applications beyond resolution level three.

In the present work we overcome this difficulty by developing an inverse-free recursive equivalent formulation of the multiresolution algorithms. This new formulation allows us to consider direct or iterative methods which take advantage of the very special sparse structure of the involved matrices for solving the associated linear systems.

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The rest of this document is organized as follows: In Section 2, we present the required basic concepts concerning the considered data approximation problem. In Section 3, we present the standard recursive multiresolution algorithms: decomposition and reconstruction. In Section 4, we present and analyze the proposed inverse-free recursive multiresolution algorithms. We also include the results of some numerical experiments for two different applications: high energy detection and noise reduction. Finally, in Section 5, we discuss for large-scale data approximation problems the use of the conjugate gradient iterative method coupled with a specialized preconditioner that takes advantage of the structure of the involved matrices.

2. Preliminaries and basic concepts

Let $D \subset \mathbb{R}^2$ be a polygonal domain (an open, non-empty connected set) in such a way that $\overline{D}$ admits a Δ^1-type triangulation (see e.g. [6]). Let us consider the Sobolev space $\mathcal{H}^2(D)$, whose elements are (classes of) functions u defined on D such that their partial derivatives (in the distribution sense) $\partial^\beta u$ belong to $\mathcal{L}^2(D)$, with $\beta = (\beta_1, \beta_2) \in \mathbb{N}^2$ and $1 \le |\beta| \le 2$, where $|\beta| = \beta_1 + \beta_2$. In this space we consider the usual inner semi-products

$$(u, v)_t = \sum_{|\beta|=t} \int_D \partial^\beta u(x) \partial^\beta v(x) dx \quad \text{for } t = 0, 1, 2,$$

the corresponding semi-norms

$$|u|_t = (u, u)_t^{1/2} = \left(\sum_{|\beta|=t} \int_D \partial^\beta u(x)^2 dx \right)^{1/2} \quad \text{for } t = 0, 1, 2,$$

and the norm

$$\|u\| = \left(\sum_{t=0}^{2} |u|_t^2 \right)^{1/2}.$$

We will consider the Powell–Sabin subtriangulation $\mathcal{T}_6$ associated to $\mathcal{T}$ (see [7] or [8]), which is obtained by subdividing each triangle $T \in \mathcal{T}$ into six subtriangles by connecting each vertex of T to the midpoint of the opposite side. Hence, all these micro triangles have the barycenter of T as a common vertex. Nevertheless, Powell–Sabin subtriangulation can be also obtained by using the incenter of T instead of the barycenter in the split procedure for a larger class of triangulations (see [9]).

Let us consider the set

$$\mathcal{S}_2^1(D, \mathcal{T}_6) = \{v \in \mathcal{C}^1(D) : v|_{T'} \in \mathbb{P}_2(T')\ \forall T' \in \mathcal{T}_6\},$$

where $\mathbb{P}_2(T')$ stands for the space of bivariate polynomials of total degree at most two over T'. In [7] it is shown that given a function $f \in \mathcal{C}^m(\overline{D})$, $m \ge 1$, there exists a unique function $v \in \mathcal{S}_2^1(D, \mathcal{T}_6)$ such that the values of v and all its first partial derivatives coincide with those of f at all the vertices of $\mathcal{T}$.

Let us now consider a finite subset $\mathcal{P} = \{p_1, \dots, p_q\}$ of points in $\overline{D}$ and a given vector of real values $Z = (z_i)_{i=1}^q \in \mathbb{R}^q$. From the continuous injection of $\mathcal{H}^2(D)$ into $\mathcal{C}^0(\overline{D})$, we can define the evaluation operator

$$\begin{array}{rcl} \rho : \mathcal{H}^2(D) & \longrightarrow & \mathbb{R}^q \\ v & \mapsto & \rho(v) = (v(p_i))_{i=1}^q. \end{array}$$

We are looking for a $\mathcal{C}^1$-surface that approximates the points $\{(p_i, z_i)\}_{i=1}^q \subset \mathbb{R}^3$ by minimizing the functional energy

$$\mathcal{J}(v) = \langle \rho(v) - Z \rangle_q^2 + \tau_1 |v|_1^2 + \tau_2 |v|_2^2, \tag{1}$$

where $\langle \cdot \rangle_q$ represents the usual Euclidean norm in $\mathbb{R}^q$, $\tau_1 > 0$ and $\tau_2 \ge 0$. Observe that the first term of $\mathcal{J}$ measures (in the least squares sense) how well v approximates the values in Z, while the second one represents the "minimal energy condition" over the semi-norms $|\cdot|_1$, and $|\cdot|_2$ weighted by the parameters τ_1 and τ_2, respectively. More precisely the minimization problem under consideration is:

Problem 1. Find $\sigma \in \mathcal{S}_2^1(D, \mathcal{T}_6)$ such that $\mathcal{J}(\sigma) \le \mathcal{J}(v)$ for all $v \in \mathcal{S}_2^1(D, \mathcal{T}_6)$.

In [2] it is shown that, under the condition $\mathrm{Ker}(\rho) \cap \mathbb{P}_1(D) = \{0\}$, Problem 1 has a unique solution. Moreover, if $N = \dim(\mathcal{S}_2^1(D, \mathcal{T}_6))$, we consider a basis $\{v_1, \dots, v_N\}$ of the space $\mathcal{S}_2^1(D, \mathcal{T}_6)$, and we denote the unique solution of Problem 1 as $\sigma = \sum_{i=1}^N \alpha_i v_i$, then the coefficients vector $(\alpha_i)_{i=1}^N$ is obtained as the unique solution of the linear system

$$CX = T, \tag{2}$$

where

$$C = \left(\langle \rho(v_i), \rho(v_j) \rangle_q + \sum_{t=1}^{2} \tau_t (v_i, v_j)_t \right)_{i,j=1}^N \quad \text{and} \quad T = \left((\langle Z, \rho(v_i) \rangle_q)_{i=1}^N \right)^T,$$

where $\cdot^T$ denotes the transposition operation. In addition, in [2] it is also shown that the coefficient matrix C is banded, symmetric and positive definite.