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Derivation and analysis of Lattice Boltzmann schemes for the linearized Euler equations

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ABSTRACT

We derive Lattice Boltzmann (LBM) schemes to solve the Linearized Euler Equations in 1D, 2D, and 3D with the future goal of coupling them to an LBM scheme for Navier Stokes Equations and a Finite Volume scheme for Linearized Euler Equations. The derivation uses the analytical Maxwellian in a BGK model. In this way, we are able to obtain second-order schemes. In addition, we perform an L^2-stability analysis. Numerical results validate the approach.

1. Introduction

In the field of computational aeroacoustics (CAA) techniques from computational fluid dynamics are used to predict aeroacoustic phenomena. In a project with an industry partner, we are investigating the aeroacoustic far-field generated by highly vortical flows streaming through a flat plate silencer built from porous media. Hasert [1] identified three different length-scales within this setting: 1. the size of pores within the porous medium ($\mathcal{O}(10\ \mu\text{m})$); 2. the length of the vortical flow ($\mathcal{O}$ (mm)); and 3. the dimension of the inviscid acoustic far-field ($\mathcal{O}$ (m)). In order to resolve the acoustic effects of the porous medium properly, the Lattice Boltzmann method (LBM) is used to solve the Navier Stokes Equation (NSE). Since within the acoustic far-field viscosity can be neglected, we simulate the far-field using the Linearized Euler Equation (LEE). We plan to use Finite Volume methods (FVM) for solving the LEE, due to their conservation form and the large length-scale of the acoustic far-field. The latter point is important for keeping computational complexity manageable. In order to follow this approach, a coupling of the kinetic LBM for the NSE and the macroscopic FVM for the LEE is necessary.

This coupling itself introduces two difficulties. First, we need to couple the viscous NSE with the inviscid LEE. Here, the problem is that we expect an abrupt change between those models to introduce non-physical effects such as spurious reflections of sound waves. We plan to use a smooth transition model similar to [2,3] smearing the change in viscosity over a buffer zone in order to minimize these non-physical effects. Second, we need to couple the mesoscopic LBM with the macroscopic FVM. Here, the translation from mesoscopic particle density functions to macroscopic quantities can be done using moments while the other way round is non-trivial. Again, we are especially interested in avoiding non-physical effects such as the above mentioned spurious reflections of sound waves. In order to be able to handle both problems separately, we decided to split this coupling of both model and method into two steps: 1. switch the model from NSE to LEE; and 2. switch the method from LBM to FVM. This provides us with the possibility to separate the derivation of a smooth transition model for the change from NSE to LEE from the choice of appropriate translations of macroscopic to mesoscopic quantities. By this, we

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gain the freedom to choose approaches ideal for each problem. The drawback of this two-step approach is that it introduces a buffer-zone in which we need to solve the LEE using LBM. In this paper we present an intermediate step in the derivation of the needed LBM of the LEE used in the aforementioned buffer-zone: LBM schemes for the LEE without background velocity. Research on how to extend these LBM schemes to incorporate background velocities is already undergoing.

During the last 20 years, lots of research on the LBM were undertaken. He and Luo [4] showed that the LBM is not only a generalization of Lattice Gas Automata (LGA) but is a discretization of the Boltzmann Bhatnagar–Gross–Krook (BGK) equation. In [5] Junk et al. performed a rigorous analysis of the LBM using well-established tools from analysis of Finite Difference Methods (FDM). For the problem of stability of LBM, different approaches were investigated: 1. direct von Neumann analysis of the linearized LBM (Sterling and Chen [6], Lallemand and Luo [7]); 2. entropic LBM with equilibrium distributions that admit an H-theorem (Chen and Teixeira [8], Karlin et al. [9]); 3. rigorous stability analysis with respect to a weighted L^2-norm (Banda et al. [10], Junk and Yong [11], Junk and Yang [12]); and 4. application of concepts from nonequilibrium thermodynamics (Yong [13]). Bernsdorf et al. [14] first showed the suitability of the LBM for flows through complex geometries. In addition, Buick et al. [15], Dellar [16], Crouse et al. [17], Lallemand and Luo [7,18], and Marie et al. [19] analyzed, tested, and discussed the LBM as a tool for acoustic simulations. Based on the capability of the LBM for complex geometries and acoustics, Hasert et al. [20] and Hasert [1] used the LBM enhanced by a sub-grid model to resolve the acoustic field generated by a flow through a porous medium. By using the LBM, they were able to directly simulate the aeroacoustic contributions of the individual pores. Independent of this, in the field of continuous analysis of the Boltzmann equation Bardos et al. [21] derived the Linearized Euler equations for monatomic gases as limit of the continuous Boltzmann equation in acoustic scaling. This paper was the starting point of the derivations presented in this paper. The use of LEE for CAA offers a resource-saving alternative over the classical use of NSE due to their simpler, linear structure. Mankbadi et al. [22] used the LEE for simulation of supersonic jet noise. Further studies of the acoustic capabilities of the LEE were for example taken out by Bailly and Juvé [23] and Bogey et al. [24]. Roller et al. [25] showed that in a hybrid approach the LEE are a well-suited and efficient method for simulating the acoustic far-field.

This work is structured as follows. First, in Section 2 we briefly present the Linearized Euler Equations without background velocity as used in this paper and briefly recapitulate the basics of the Boltzmann Equation. In Section 3, we adapt the results by Bardos et al. [21] to the LEE for monatomic gases as used in this paper. Based on this, we then derive the semi-discrete Finite Discrete Velocity Models for monatomic gases in Section 4 and generalize these results to polyatomic gases in Section 5. In these sections, we will also post necessary conditions on the velocity models. In Sections 6 and 7, we then present the fully discrete Lattice Boltzmann Equation and perform an analysis of consistency and stability of this equation. Based on the necessary conditions derived in Sections 4 and 5, in Section 8 we then present velocity models for monatomic and diatomic gases respectively. In addition, stability of the LBM for these velocity models is analyzed using the results from Section 7. Then, numerical results are presented and discussed in Section 9. Finally, in Section 10 we wrap up the results of this paper.

2. Theoretical background

2.1. The linearized Euler equations

In the field of Fluid Dynamics the compressible Euler Equations describe inviscid flows. In settings in which the fluid flow is dominated by a constant background flow the Euler Equations can be linearized around this flow to reduce their complexity. Assume a constant background flow with density ρ_0, temperature θ_0, and velocity $\boldsymbol{u}_0$. Linearization of the Euler Equations around this background flow then yields the Linearized Euler Equations (LEE). In this setting the macroscopic variables of the fluid density ρ, velocity $\boldsymbol{u}$, temperature θ, and pressure p are given as:

$$\rho = \rho_0 + \epsilon\rho', \tag{1a}$$

$$u = \boldsymbol{u}_0 + \epsilon\boldsymbol{u}', \tag{1b}$$

$$\theta = \theta_0 + \epsilon\theta', \tag{1c}$$

$$p = p_0 + \epsilon p'. \tag{1d}$$

where the primed variables represent fluctuations around the background flow and the parameter ϵ represents the scale of the fluctuations. The velocities $\boldsymbol{u}$ and $\boldsymbol{u}'$ are $\boldsymbol{u} = u_x$ and $\boldsymbol{u}' = \boldsymbol{u}'_x$ respectively in the 1D case, $\boldsymbol{u} = (u_x, u_y)^T$ and $\boldsymbol{u}' = (u'_x, u'_y)^T$ respectively in the 2D case, and $\boldsymbol{u} = (u_x, u_y, u_z)^T$ and $\boldsymbol{u}' = (u'_x, u'_y, u'_z)^T$ respectively in the 3D case. The temperature θ used throughout this paper does not describe the temperature T in Kelvin but the scaled temperature $\theta = TR_{specific}$ where $R_{specific}$ denotes the specific gas constant. Since the pressure in an ideal gas is given by $p = \rho\theta$, for the fluctuations in pressure we have $p' = \rho_0\theta' + \theta_0\rho'$.

As already stated in the introduction, we focus on the LEE for flows without background velocity, i.e. $\boldsymbol{u}_0 = \boldsymbol{0}$. This can either be the case if no background velocity is present, its magnitude is small enough to incorporate it into the fluctuations $\boldsymbol{u}'$, or by choice of an appropriate Galilean frame. The assumption of a such a Galilean frame can be easily justified for the continuous case. Under certain conditions, one might be able to implement such a Galilean frame for boundary-free problems