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The lowest order characteristic mixed finite element scheme for convection-dominated diffusion problem

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ABSTRACT

A new characteristic mixed finite element method (MFEM) is proposed by combining the method of characteristics with the new mixed variational formulation for the convection-dominated diffusion problem. The error estimates for both the original variable u and auxiliary variable $\vec{p}$ are derived via the properties of the integral identity and mean value techniques. At the same time, some numerical experiments are provided to confirm validity of the theoretical analysis and excellent performance of the proposed method.

1. Introduction

Considering the following convection-dominated diffusion problem:

$$\begin{cases} u_t + \mathbf{a}(X,t)\cdot\nabla u - \nabla\cdot(b(X)\nabla u) = f(X,t), & (X,t) \in \Omega\times(0,T), \\ u(X,t) = 0, & (X,t)\in \partial\Omega\times(0,T), \\ u(X,0) = u_0(X), & X\in \Omega, \end{cases} \tag{1.1}$$

where $X = (x, y)$, $\Omega \subset R^2$ is a bounded convex polygonal domain with Lipschitz continuous boundary, $T \in (0, +\infty)$ is a set value and $u_0(X)$ is a given smooth function. ∇ and $\nabla\cdot$ denote the gradient and the divergence operator, respectively.

The parameters appearing in (1.1) satisfy the following assumptions [1]:

(i) $\mathbf{a}(X,t) = (a_1(X,t), a_2(X,t))$ represents fluid velocity satisfying

$$\nabla\cdot\mathbf{a}(X,t) = 0, \qquad |\mathbf{a}(X,t)| \le C, \quad \forall X\in\Omega, \tag{1.2}$$

here C is a constant;

(ii) $b(X)$ is sufficiently smooth and there exist constants b_1 and b_2, such that

$$0 < b_1 \le b(X) \le b_2 < +\infty, \quad \forall X\in\Omega. \tag{1.3}$$

The convection-dominated diffusion problem is of vital importance in many applications and disciplines, it describes such phenomena as the flow of heat within a moving fluid, the movement of aerosols and trace gases in the atmosphere, the incompressible miscible displacement in porous media and so on. Solving such convection-dominated diffusion problems numerically is challenging, the standard finite difference method (FDM) or finite element method (FEM) does not perform well as it often exhibits sharp fronts and excessive nonphysical oscillations. In order to overcome these difficulties, many effective

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schemes have been proposed for this equation, such as streamline diffusion method [2,3], Eulerian–Lagrangian method [4,5], least-squares-mixed FEM [6,7], the modified characteristics-Galerkin FEM [8,9], adaptive Galerkin-characteristic method [10], characteristic nonconforming FEM [11–14], characteristic MFEM [15,16] and expanded characteristic MFEM [1,17], characteristic FDM [18,19], characteristic-finite volume element method [20,21] and characteristic-mixed covolume methods [22], and the integral equation formulations based on boundary element method [23,24]. The modified characteristic Galerkin FEM is a combination of characteristics method and standard Galerkin FEM. In this scheme, the time derivative and the advection term are combined as a directional derivative along the characteristics, which makes the resulting matrix symmetric and reduces the numerical dispersion and oscillation. In addition, it permits large time steps in a simulation without loss of accuracy (cf. [1,8]).

Recently, a new mixed variational form for second elliptic problem was presented in [25,26]. It has a significant advantage: the LBB condition is automatically satisfied when the gradient of approximation space for the original variable is included in approximation space for the flux variable. Subsequently, this method was further applied to parabolic problems [27], Sobolev equations [28], convection diffusion problems [29], etc. The purpose of this article is to combine the above new MFE scheme with the method of characteristics to establish the lowest order characteristic MFE scheme for (1.1) (the bilinear element is used for approximating the original variable u and the auxiliary variable $\vec{p}$ is approximated by the lowest order Nédélec element space), the total degree of freedom of this finite element pair is only $3NP$ which is $2NP$ less than that of in [29] (NP denotes the total number of node points of the partition). By use of some properties of the interpolation operator, the integral identity and mean value techniques, the error estimates of $O(h^2)$ order for u in L^2-norm, and $O(h)$ order for u in H^1-norm and $\vec{p}$ in L^2-norm are derived, respectively.

The outline of this paper is as follows. Section 2 is devoted to introduce the MFE approximation spaces and give the construction of the new characteristic MFE scheme. In Section 3, the convergence analysis and error estimates for both the original variable u and the auxiliary variable $\vec{p}$ are obtained. In Section 4, some numerical examples are provided to illustrate the effectiveness of our proposed method. Finally conclusions are given in Section 5.

2. The MFEs and the new characteristic MFE scheme

Let $W^{k,p}(\Omega)$ be the standard Sobolev space with the norm

$$\|v\|_{W^{k,p}(\Omega)} = \begin{cases} \Big(\sum\limits_{|m|\le k}\int_\Omega |D^m v|^p \mathrm{d}X\Big)^{\frac{1}{p}}, & 1 \le p < +\infty, \\ \max\limits_{|m|\le k} \operatorname*{ess\,sup}\limits_{x\in\Omega} |D^m v|, & p = +\infty. \end{cases}$$

and $H^k(\Omega) = W^{k,2}(\Omega)$, $H^0(\Omega) = L^2(\Omega)$, where $dX = dxdy$.

Then we introduce Sobolev space involving time [30]

$$W^{k,p}(t_1, t_2; \Phi) = \Big\{ v : \Big\| \frac{\partial^m v}{\partial t^m}(\cdot, t) \Big\|_\Phi \in L^p(t_1, t_2), 0 \le m \le k, 1 \le p \le +\infty \Big\}$$

equipped with the norm

$$\|v\|_{W^{k,p}(t_1,t_2;\Phi)} = \begin{cases} \Big(\sum\limits_{m=0}^{k}\int_{t_1}^{t_2} \Big\|\frac{\partial^m v}{\partial t^m}\Big\|_\Phi^p \mathrm{d}t\Big)^{\frac{1}{p}}, & 1 \le p < +\infty, \\ \max\limits_{0\le m\le k} \operatorname*{ess\,sup}\limits_{t\in[t_1,t_2]} \Big\|\frac{\partial^m v}{\partial t^m}(\cdot, t)\Big\|_\Phi, & p = +\infty. \end{cases}$$

Assume that the domain Ω be a polygon domain with edges parallel to the coordinate axes, T_h be a rectangular subdivision of Ω, which does not need to satisfy the regular condition [31].

For a given element $e \in T_h$, let $e = [x_e - h_{x_e}, x_e + h_{x_e}] \times [y_e - h_{y_e}, y_e + h_{y_e}]$, where (x_e, y_e) is the barycenter of e and $2h_{x_e}$, $2h_{y_e}$ are the length of edges parallel to x-axis and y-axis, respectively. The four vertices $d_1 = (x_e - h_{x_e}, y_e - h_{y_e})$, $d_2 = (x_e + h_{x_e}, y_e - h_{y_e})$, $d_3 = (x_e + h_{x_e}, y_e + h_{y_e})$ and $d_4 = (x_e - h_{x_e}, y_e + h_{y_e})$, the four edges $l_i = \overline{d_i d_{i+1}}(\text{mod } 4)(i = 1, 2, 3, 4)$, $h_e = \max_{e\in T_h}\{h_{x_e}, h_{y_e}\}$, $h = \max_{e\in T_h} h_e$.

Then the bilinear element space M_h and the lowest order Nédélec element space $\mathbf{V}_h$ are defined as [32]

$$M_h = \Big\{ v_h : v_h|_e \in Q_{1,1}(e), v_h|_{\partial\Omega} = 0 \Big\}$$

and

$$\mathbf{V}_h = \big\{ \vec{w}_h = (w_{1h}, w_{2h}) : \vec{w}_h|_e \in Q_{0,1}(e) \times Q_{1,0}(e) \big\},$$

respectively, where $Q_{i,j} = span\{x^i y^j\}$, $0 \le i, j \le 1$.

We denote this finite element pair by $Q_{1,1} - Q_{0,1} \times Q_{1,0}$.