Computers & Operations Research 63 (2015) 161–171

Contents lists available at ScienceDirect

Computers & Operations Research

journal homepage: www.elsevier.com/locate/caor

Models and algorithms for packing rectangles into the smallest square

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ARTICLE INFO

Available online 14 May 2015

Keywords:
Two dimensional packing
Mathematical model
Randomized algorithm

ABSTRACT

We consider the problem of determining the smallest square into which a given set of rectangular items can be packed without overlapping. We present an ILP model, an exact approach based on the iterated execution of a two-dimensional packing algorithm, and a randomized metaheuristic. Such approaches are valid both for the case where the rectangles have fixed orientation and the case where they can be rotated by 90°. We computationally evaluate the performance and the limits of the proposed approaches on a large set of instances, including a number of classical benchmarks from the literature, for both cases above, and for the special case where the items are squares.

1. Introduction

We consider the problem of packing a given set of rectangular *items* into a *square bin* of minimum edge. The items must be packed, without overlapping, with their edges parallel to the edges of the bin. Using the typology of cutting and packing problems proposed by Waescher et al. [31], this problem belongs to the "open dimension" class, characterized as

[*input minimization/variable dimension/arbitrary assortment of small rectangular items*].

We will address the following specific versions of the problem, as classified by Caprara et al. [8]:

PRSO (*Packing Rectangles into a Square in the Oriented case*): the items are oriented rectangles, i.e., they cannot be rotated;
PRSR (*Packing Rectangles into a Square with Rotation*): the items are rectangles that can be rotated by 90°,

plus the special case of both:

PSS (*Packing Squares into a Square*): the items are squares.

Packing into squares has a number of practical applications, for example in VLSI design when a set of rectangular components has to be packed on a square chip. In addition, these problems can arise as subproblems to be solved in more complex two-dimensional packing problems like, e.g., those arising when rectangular objects that cannot be stacked have to be loaded on square pallets.

The problems we consider have been treated in the literature mainly from a theoretical point of view. Problem PSS was proved to be strongly NP-hard by Leung et al. [23], hence the same holds for PRSO and PRSR. Caprara et al.[8] introduced some simple lower bounds for the three problem versions, and analyzed their worst-case performance. Bansal et al. [3] gave a polynomial time approximation scheme (PTAS) for problem PRSO. The result was extended to PRSR by Correa [12]. These results immediately lead to a PTAS for PSS.

From a computational point of view, the running time of the above mentioned algorithms (which is exponential in the inverse of the relative error) rules out the possibility of their use for the practical solution of these problems. Picouleau [29] proposed simple heuristics for PSS, and determined the worst-case performance of some of them.

We are not aware of other contributions on the specific problems we are dealing with. Different, but similar, open dimension problems have been addressed by Korf [20,21], Huang and Korf [17] and Korf et al. [22], who considered the problem of packing a given set of rectangles into a rectangle of minimum area. This problem appears to be more challenging than that of packing into the smallest square, as the two sides of the optimal bin have to be determined instead of one.

More different open dimension problems have been also studied, concerning, e.g., the packing of cylinders (Birgin et al. [7]), of polygons (Stoyan and Patsuk [30]) or of irregular shapes (Costa et al. [13]).

Other related problems belong to the area of two-dimensional packing: the two-dimensional bin packing problem will be used as a subproblem in the solution approach described in Section 3.3. An

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http://dx.doi.org/10.1016/j.cor.2015.04.024
0305-0548/

extensive literature is available for these topics, for which we refer the interested reader to the survey by Lodi et al. [24].

In this paper we propose two exact methods and a heuristic approach to the solution of problems PRSO, PRSR, and PSS. In Section 2 we provide mathematical formulations. In Section 3 we introduce a constructive heuristic, simple ways to compute lower bounds, and two exact approaches. The performance of these methods is evaluated through computational experiments on a benchmark composed by instances from the literature and randomly generated instances. In Section 4 we propose a randomized metaheuristic algorithm which has reminiscences of different metaheuristic paradigms. Further computational experiments are used to evaluate the algorithm's performance. Conclusions are drawn in Section 5.

Throughout the paper we assume, without loss of generality, that all input data are positive integers.

2. Integer linear models

In this section we provide Integer Linear Programming (ILP) models for problems PRSO and PRSR introduced in Section 1, problem PSS being a special case of both. We follow the modeling approach commonly used for two-dimensional packing problems (see Beasley [5]), which requires a pseudo-polynomial number of variables and constraints. A different modeling technique, based on the enumeration of all possible relative placements of each pair of items, was proposed by Onodera et al. [28] for a special two-dimensional block placement problem. Their method can lead to models of polynomial size, but with known low efficiency in practice (see, e.g., Chen et al. [10]), so the research in this area mostly concentrated on the former approach (see, e.g., Hadjiconstantinou and Christofides [16]).

The following definitions are common to the three considered problems. The input consists of n rectangular items, each characterized by width w_j and height h_j $(j=1,\dots,n)$, to be packed without overlapping into the smallest possible square of side z. Let u be any upper bound on the optimal solution value (edge of the enclosing square bin).

Assume that the bottom-left corner of the bin is at coordinate $(0,0)$ of a Cartesian system having axes x and y and integer coordinates. The problem of packing rectangles into a square in the oriented case (PRSO) can then be modeled by introducing binary variables

$$\xi^j_{pq} = \begin{cases} 1 & \text{if item } j \text{ is packed (without rotation) with its bottom}-\text{left corner at } (p,q); \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

for $j=1,\dots,n$, $p=0,\dots,u-w_j$, $q=0,\dots,u-h_j$, where coordinates $x>u-w_j$ (resp. $y>u-h_j$) are excluded for obvious reasons.

In order to reduce the number of binary variables, we can make use of a consideration made by Christofides and Whitlock [11] for the two-dimensional knapsack problem. Observe indeed that, for any feasible packing in a square of edge u, there exists an equivalent packing in a square of edge not greater than u in which each item j is moved left and then down as much as possible. In this way, each item will have both its left and bottom edge touching either the enclosing square or another item. We can consequently define a set of *normal coordinates*, i.e., coordinates that can be obtained by combination of all widths (resp. heights) of the other items, namely

$$W_j = \left\{ x : 0 \le x \le u-w_j, x = \sum_{k \in S} w_k \quad \text{for some } S \subseteq \{1,\dots,n\}\backslash\{j\} \right\};$$

$$H_j = \left\{ y : 0 \le y \le u-h_j, y = \sum_{k \in S} h_k \quad \text{for some } S \subseteq \{1,\dots,n\}\backslash\{j\} \right\}.$$

In (1) we will then only define ξ^j_{pq} variables for which $p \in W_j$ and $q \in H_j$ $(j=1,\dots,n)$. The resulting model for PRSO is then

$$\min z \tag{2}$$

$$\sum_{p \in W_j} \sum_{q \in H_j} \xi^j_{pq} = 1 \quad (j=1,\dots,n) \tag{3}$$

$$\sum_{j=1}^{n} \sum_{\substack{p=r-(w_j-1) \\ p \in W_j}}^{r} \sum_{\substack{q=s-(h_j-1) \\ q \in H_j}}^{s} \xi^j_{pq} \le 1 \qquad (r,s=0,\dots,u-1) \tag{4}$$

$$z \ge \sum_{p \in W_j} \sum_{q \in H_j} p\, \xi^j_{pq} + w_j \qquad (j=1,\dots,n) \tag{5}$$

$$z \ge \sum_{p \in W_j} \sum_{q \in H_j} q\, \xi^j_{pq} + h_j \qquad (j=1,\dots,n) \tag{6}$$

$$\xi^j_{pq} \in \{0,1\} \qquad (j=1,\dots,n;\ p \in W_j;\ q \in H_j). \tag{7}$$

The objective function (2) minimizes the edge z of the resulting square bin. Eqs. (3) establish that the bottom-left corner of each item is placed in exactly one position. Inequalities (4) impose that any unit square, having its bottom-left corner, say, at coordinate (r,s), is occupied by at most one item. Finally, (5) and (6) define the value of z as the maximum right and upper side of an allocated item. The model has pseudo-polynomial size, as it requires u^2+3n constraints and $\sum_{j=1}^{n} |W_j| \cdot |H_j|$, i.e., $O(nu^2)$, variables. As it will be evident from the computational analysis of Section 3.4, such size can be very large in practice, when the number of items and/or their size are big.

The ILP model for PRSO is obviously valid for PSS as well. In this case the model can be simplified by observing that $W_j = H_j$ for all j.

An ILP model for PRSR can be derived from the previous one as follows. In addition to variables ξ^j_{pq}, see (1), let

$$\vartheta^j_{pq} = \begin{cases} 1 & \text{if item } j \text{ is packed, rotated by } 90^\circ, \text{ with its bottom}-\text{left corner at } (p,q); \\ 0 & \text{otherwise} \end{cases} \tag{8}$$

for $j=1,\dots,n$, $p=0,\dots,u-\min(w_j,h_j)$, $q=0,\dots,u-\min(w_j,h_j)$. The number of variables can again be reduced by only considering normal coordinates. In this case however, for each item j, the (unique) set E_j of normal coordinates (to be used both for x and y) includes all values that can be obtained by combination of all widths *and* heights of the other items, namely

$$E_j = \left\{ v : 0 \le v \le u-\min(w_j,h_j), \quad v = \sum_{k \in S} w_k + \sum_{k \in T} h_k; \quad S,T \subseteq \{1,\dots,n\}\backslash\{j\}, S \cap T = \varnothing \right\}.$$

We obtain

$$\min z \tag{9}$$

$$\sum_{p,q \in E_j} (\xi^j_{pq} + \vartheta^j_{pq}) = 1 \qquad (j=1,\dots,n) \tag{10}$$

$$\sum_{j=1}^{n} \left(\sum_{\substack{p=r-(w_j-1) \\ p \in E_j}}^{r} \sum_{\substack{q=s-(h_j-1) \\ q \in E_j}}^{s} \xi^j_{pq} + \sum_{\substack{p=r-(h_j-1) \\ p \in E_j}}^{r} \sum_{\substack{q=s-(w_j-1) \\ q \in E_j}}^{s} \vartheta^j_{pq} \right) \le 1 \quad (r,s=0,\dots,u-1) \tag{11}$$

$$z \ge \sum_{p,q \in E_j} p\, \xi^j_{pq} + w_j \quad (j=1,\dots,n) \tag{12}$$

$$z \ge \sum_{p,q \in E_j} p\, \vartheta^j_{pq} + h_j \quad (j=1,\dots,n) \tag{13}$$