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Representation of the non-dominated set in biobjective discrete optimization

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ABSTRACT

This paper introduces several algorithms for finding a representative subset of the non-dominated point set of a biobjective discrete optimization problem with respect to uniformity, coverage and the ϵ-indicator. We consider the representation problem itself as multiobjective, trying to find a good compromise between these quality measures. These representation problems are formulated as particular facility location problems with a special location structure, which allows for polynomial-time algorithms in the biobjective case based on the principles of dynamic programming and threshold approaches. In addition, we show that several multiobjective variants of these representation problems are also solvable in polynomial time. Computational results obtained by these approaches on a wide range of randomly generated point sets are presented and discussed.

1. Introduction

Typically, multiobjective optimization problems are solved according to the Pareto principle of optimality: A solution is called *efficient* if there is no other feasible solution which is at least as good in all objectives and strictly better in at least one of them. Each of these efficient solutions corresponds to a good compromise among a number of alternatives, and each of them is potentially relevant to a decision maker. Therefore, the goal of multiobjective optimization is to compute the efficient set, from which the decision maker chooses the most preferable solution. Since this set may be too large to present to a decision maker, procedures that produce succinct representations of the efficient set are of particular interest in the context of practical applications.

This paper focuses on the computation of a *good* representation of the efficient set, the so-called *representation problem*, where the quality of the representation is measured with respect to some property of interest [1,2]. The three underlying assumptions are: (i) the efficient set is given; (ii) the decision maker is able to choose the preferred solution based solely on its location in the objective space (an efficient solution corresponds to a *non-dominated point* in the objective space); and (iii) the cardinality of the representation (k) is provided. Two widely accepted ways of measuring the quality of a representation [3–6] were introduced by Sayin [2], and are explored in this paper: (i) *uniformity*, the representation points are as spread as possible and (ii) *coverage*, the representation points are close to the remaining non-dominated points. As these two quality measures may be conflicting, the representation problem that arises from the combination of these two properties, as two objectives, is also of interest.

A third quality measure, known as the ϵ-*indicator* [7], is also considered in this paper. The ϵ-indicator is a widely accepted measure of performance of heuristic approaches to multiobjective optimization. In this paper, this indicator is used as a measure of the quality of the representation, and is shown to be closely related to the notion of coverage.

This paper considers only representation problems for biobjective combinatorial optimization problems, for which polynomial-time algorithms can be derived for all three quality measures. Since the set of all non-dominated points can be totally ordered in the biobjective case, the representation problem can be recast as a special type of one-dimensional facility-location problems, such as the k-center and k-dispersion problem, for which certain properties can be efficiently explored from an algorithmic point of view.

The three quality measures considered give rise to different representation problems with bottleneck objective functions. Bottleneck objective functions [8] represent a class of objective functions whose goal is to maximize (over all subsets) a minimum quantity over all the elements, or vice versa. The algorithms proposed in this paper fall into two typical types of approaches

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Table 1
Time-complexities of the dynamic programming and threshold algorithm for the considered representation problems.

Problem	Dynamic programming	Threshold algorithm
Uniformity	$\mathcal{O}(k\,n+n\log n)$	$\mathcal{O}(n^2\log n)$
Coverage	$\mathcal{O}(k\,n+n\log n)$	$\mathcal{O}(n^2\log n)$
ϵ-Indicator	$\mathcal{O}(k\,n+n\log n)$	$\mathcal{O}(n^2\log n)$
Coverage–uniformity	$\mathcal{O}(k\,n^4\log n)$	$\mathcal{O}(n^4)$
ϵ-Indicator–uniformity	$\mathcal{O}(k\,n^4\log n)$	$\mathcal{O}(n^4)$
Coverage–ϵ-indicator	$\mathcal{O}(k\,n^4\log n)$	$\mathcal{O}(n^3\log n)$
Coverage–ϵ-indicator–uniformity	$\mathcal{O}(k\,n^6\log n)$	$\mathcal{O}(n^6)$

to this class of problems: dynamic programming and threshold approaches. Whereas dynamic programming consists of solving a sequence of smaller optimization problems, threshold approaches solve a sequence of related feasibility problems. The various algorithms are designed so that they can be easily integrated to solve the representation problems that arise when two or more quality measures are considered together. Table 1 shows the time-complexities that were achieved with the two approaches described in this paper for solving several representation problems, where n is the cardinality of the non-dominated set and k is the cardinality of the representation.

The paper is organized as follows. Section 2 introduces definitions and notation that will be used throughout the paper. Sections 3, 4 and 5 introduce the three representation problems according to uniformity, coverage and the ϵ-indicator, respectively, as well as the two solution methods and experimental results. Then, Section 6 introduces four multiobjective formulations of the representation problem. Finally, Section 7 presents a discussion and conclusions about the main results of this paper.

2. Definitions and notation

In this section, we describe optimality concepts of biobjective optimization problems, as well as several formulations of representation problems that arise from three proposed representation quality measures: uniformity, coverage and ϵ-indicator.

2.1. Optimality concepts

We consider biobjective, discrete optimization problems with two maximizing objectives, i.e.,

$$\operatorname*{vmax}_{x\in X} f(x) = \operatorname*{vmax}_{x\in X}\left(f^1(x), f^2(x)\right) \tag{1}$$

where X denotes the set of feasible solutions and $f^i : X \to \mathbb{R}$, $i=1,2$, are two (generally conflicting) objective functions. Let $x, x' \in X$. In the context of Pareto optimality, we introduce the following dominance relations [9]:

- $f(x) \geq f(x')$, i.e., $f(x)$ *weakly dominates* $f(x')$, if and only if $f^i(x) \geq f^i(x'), i=1,2$;
- $f(x) > f(x')$, i.e., $f(x)$ *dominates* $f(x')$, if and only if $f(x) \geq f(x')$ and $f(x) \neq f(x')$.

These definitions immediately generalize to higher dimensional problems and to minimization problems. When neither $f(x) > f(x')$ nor $f(x') > f(x)$ holds, we say that the objective vectors $f(x)$ and $f(x')$ are *incomparable*. We also use the same notation among solutions, when the corresponding relation holds in the objective function space. A solution $x \in X$ is called *efficient* (or *Pareto optimal*), if and only if there is no other feasible solution $x' \in X$ such that $f(x') > f(x)$; in this case its corresponding objective vector is called *non-dominated*. The set of all efficient solutions is called the *efficient set* and the set of all non-dominated vectors is called the *non-dominated set* (represented as $\text{vmax}_{x\in X} f(x)$ in Eq. (1)). Note that the vectors in the non-dominated set are pairwise incomparable.

Throughout this paper we assume that the feasible set X and its image $Y = f(X) \subset \mathbb{R}^2$ are discrete, and that the non-dominated set $B \subseteq Y$ is finite. We search for a good representation $R \subseteq B$ of the non-dominated set, where different quality measures are used to distinguish between different representations. In the following sections, we assume that the non-dominated set B is known and that a positive integer k with $k \leq n = |B|$ is given. According to some measure of quality, we try to find a *representative* subset $R \subseteq B$ with $|R| = k$. Without loss of generality we assume that all points in B have only positive components.

2.2. Uniformity

Sayin [2] proposed the property of *uniformity* to measure how far apart the k chosen elements of a set R are from each other, or how well they are spread over the non-dominated set. It is computed as the minimum distance between a pair of distinct elements as

$$I_U(R) = \min_{\substack{r_i, r_j \in R \\ r_i \neq r_j}} \|r_i - r_j\|, \tag{2}$$

where $\|r_i - r_j\|$ is a p-norm with $1 \leq p \leq \infty$. The goal of the *uniformity representation problem* is to find a subset R, with a given cardinality k, from a set B that maximizes $I_U(R)$, i.e.,

$$\max_{\substack{R\subseteq B\\ |R|=k}} I_U(R). \tag{UR}$$

Note that this problem corresponds to a particular case of the k-dispersion problem in facility-location; see, for example, Ravi et al. [10].

2.3. Coverage

A second property proposed by Sayin [2] is *coverage*, which measures the quality of the representative subset by considering the distance of the unchosen elements to their closest elements in the subset. Formally, the coverage of a subset R with respect to a set B is computed as

$$I_C(R,B) = \max_{b\in B}\min_{r\in R}\|r-b\|.$$

The *coverage representation problem* consists of finding the subset of cardinality k that has the smallest coverage value (coverage radius), i.e.,

$$\min_{\substack{R\subseteq B\\ |R|=k}} I_C(R,B). \tag{CR}$$

This problem is known in the literature of facility-location as the k-center problem; see Kariv and Hakimi [11] for an early reference as well as Hassin and Tamir [12] and Schöbel [13] for a problem closely related to the coverage representation problem considered here.

2.4. ϵ-indicator

The ϵ-indicator is a well-known measure of performance in heuristic solution approaches and approximation algorithms for multiobjective optimization problems [14]. In the context of the representation problem considered here, it corresponds to the smallest factor that can be multiplied to each element in the subset R such that every point in the set B becomes weakly