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A minimum expected regret model for the shortest path problem with solution-dependent probability distributions ☆

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ABSTRACT

We consider an optimization problem in which the cost of a feasible solution depends on a set of unknown parameters (scenario) that will be realized. In order to assess the cost of implementing a given solution, its performance is compared with the optimal one under each feasible scenario. The positive difference between the objective values of both solutions defines the regret corresponding to a fixed scenario. The proposed optimization model will seek for a compromise solution by minimizing the expected regret where the expectation is taken respect to a probability distribution that depends on the same solution that is being evaluated, which is called solution-dependent probability distribution. We study the optimization model obtained by applying a specific family of solution-dependent probability distributions to the shortest path problem where the unknown parameters are the arc lengths of the network. This approach can be used to generate new models for robust optimization where the degree of conservatism is calibrated by using different families of probability distributions for the unknown parameters.

1. Introduction

Optimization problems with uncertainty in the objective function or in the set of constraints have received increasing attention over the last decades due to the importance of their practical applications (see, e.g., [5–8,10,11]). Robustness Analysis and Stochastic Programming are two of the theoretical frameworks developed in Mathematical Programming to manage uncertainty or risk in order to find compromise decisions with a good behavior under any likely input data.

In the stochastic approach, it is assumed the knowledge of the probability distribution of the vector of unknown parameters. Every realization of this random vector defines a scenario associated to a deterministic instance of the optimization problem. Under this paradigm, the decision maker adopts a compromise solution by minimizing the expected cost.

In the robust optimization models, the scenarios are described without any information about how likely they can occur. The worst-case approach is one of the techniques within this framework. Its objective is to find a solution that performs reasonably well for all the scenarios. In this context, the minmax regret optimization model aims at obtaining a solution minimizing the maximum deviation (regret) between the cost of a given solution and the optimal cost under any possible scenario.

The stochastic approach is suitable when the decision maker wants to implement the obtained solution in repetitive situations. On the other hand, the robust optimization model is more conservative trying to hedge the system against the unknown scenario that may happen in specially sensitive situations needing precautionary measures (environmental interventions, public health actions, emergency contingency plans,…).

However, in some cases, the expected cost and the maximum regret can be viewed as particular instances of a family of objective functions obtained by taking the expectation according to distributions of probability that depend on the solution which is evaluated. In other words, this model introduces the possibility of using a sort of subjective probabilities that model the behavior of the uncertain vector of parameters from the point of view of the feasible solution that is considered to be implemented. The resulting approach enables us to generate new optimization models with decision criteria that reflect certain compromise between the above two perspectives and could give rise to new models used in practical situations where none of them may fit the decision maker needs (see e.g. [9]).

In the context of robust optimization, there exists a precedent to this idea and was proposed by Averbakh in [4], where a new minmax regret model is studied with a set of scenarios that is *induced* by the choice of the feasible solution. This paper considers subset-type optimization problems, where the feasible solutions

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are some subsets of a finite ground set whose elements have unknown weights. The model allows only uncertainty on those weights of the elements included in the chosen feasible solution that can take on any value of the associated uncertain intervals. On the other hand, the weights that were not included in the chosen solution, cannot deviate from their nominal values.

The paper is organized as follows, in the section two an optimization model that minimizes the expected regret is proposed. In this model, it is allowed the probability distribution for the unknown vector of parameter (scenario) may depend on the feasible solution evaluated. By modifying such a dependence it is easy to show how this model can be instantiated in other known models in the field of Stochastic Programming and Robust Optimization. In the third section a new family of probability distributions for the coefficients of the linear objective of a combinatorial optimization problem is considered. Given a feasible solution, the corresponding probability distribution for the cost scenario is fixed and a closed-form expression for the expected cost can be derived. It is also shown how to take advantage of the structure of this objective function in order to solve combinatorial problem where the feasible set is defined through linear constraints with unimodular matrix. In the last two sections, an application of this model to the shortest path in a network with uncertain arc lengths is studied and a computational experiment is presented.

2. Expected cost with solution-dependent probability distributions

Let us consider the following stochastic optimization problem:

$$\min_{x\in X} E_x f(x, \xi) \tag{P}$$

where

- f is a real-valued function,
- x is a decision vector constrained to be in the set X,
- ξ is a random vector defined on a probability space $(\Omega, \mathcal{A}, P_x)$ where the measure P_x depends on the decision vector x and
- E_x represents the expectation operator (of the random variable $f(x, \xi)$) according to the distribution generated by the probability measure P_x. $E_x f(x, \xi)$ is supposed to be finite for every $x \in X$.

The problem (P) considered here is static, that is, there are not recourse variables in different stages and X is a finite set of feasible actions (combinatorial optimization) which, in particular, implies that the minimum of the function $E_x f(x, \xi)$ exists in X. The vector ξ defines a set of parameters under which the behavior of the system is measured. Every realization of this vector will be called hereafter a scenario and it is assumed known the set of all the possible realizations, the set S.

The optimization model given by (P) has been intensively studied in the literature in those cases in which the measure of probability P_x does not depend on the decision variables x. The resulting problem is one of the central topics of the Stochastic Programming [5–7,10]. An optimal solution x^* obtained by solving this optimization model has good properties, in practice, when it is regularly implemented in the system, over and over again, under the same stochastic conditions. This could be a common situation in production planning, inventory optimization or warehouse location. In these cases, if the conditions under which the solution is implemented are independent of the previous implementations, the Strong Law of Large Numbers (see e.g. [14]) says the average of the costs converges almost sure to the optimum objective value of (P),

$$\frac{1}{n}\sum_{i=1}^{n} f(x_*, \xi^i) \xrightarrow{a.s} Ef(x_*, \xi),$$

where $\xi^1, \xi^2, \dots, \xi^n, \dots$ are independent random vectors with the same probability distribution.

Hence, once an optimal solution of the problem is regularly implemented whatever scenario may happen, the average cost converges to the optimal one. However, one should never forget that the success of this result depends on the convergence speed of the sequence of random average costs incurred by the system under this solution x^*.

In those situations where the solution will be implemented sporadically such as contingency plans, recovery planning after a disaster or, in general, solutions whose performance needs to be protected from unknown scenarios, the minmax regret optimization model could be considered. Under this paradigm, a solution whose performance is as near as possible to the optimal one under any of the considered scenarios is sought.

The expected cost and the minmax regret cost define models very different, however both of them can be integrated in a more general model under the formulation (P).

Let us define the function $z(\xi, x)$ as the cost incurred by the system when the solution $x \in X$ is implemented and the scenario ξ takes place. For example, ξ could be the coefficients of a linear cost function, $z(\xi, x) = \langle \xi, x\rangle$. The following random function measures the difference between the cost associated to a given solution and the optimal one, $z(\xi)$, under the scenario ξ,

$$f(x, \xi) = z(\xi, x) - z(\xi), \tag{1}$$

where

$$z(\xi) = \min_{x\in X} z(\xi, x). \tag{2}$$

Let us observe that, since the set X is finite, the function $z(\xi)$ is a random variable because it is the minimum of a finite number of random variables.

Taking into account the linearity of the expectation operator, it is obvious that, if $P_x = P$ for every $x \in X$ then, the problem (P) reduces to minimize the expected cost, that is,

$$\min_{x\in X} Ef(x, \xi) = \min_{x\in X}\Big[Ez(\xi, x) - Ez(\xi)\Big] = \Big[\min_{x\in X} Ez(\xi, x)\Big] - Ez(\xi),$$

where it has been assumed the existence (finiteness) of the expectation of each random variable appearing in the above chain of equations. Hence, in this case, the problem (P) coincides with the one of minimizing the mean of the cost.

On the other hand, if for each feasible solution in X, it is assumed the existence of a scenario where its objective function value has the highest difference with respect to the minimum value in the feasible set for that scenario, that is,

$$\xi(x) \in \arg\max_{\xi}[z(\xi, x) - z(\xi)], \quad \forall\, x \in X, \tag{3}$$

a probability measure P_x that concentrates all the probability mass at $\xi(x)$ can be defined, that is,

$$P_x\big[\xi = \xi(x)\big] = 1, \quad \forall\, x \in X,$$

and one has

$$\min_{x\in X} E_x f(x, \xi) = \min_{x\in X} f(x, \xi(x)) = \min_{x\in X}\max_{\xi}[z(\xi, x) - z(\xi)],$$

that is, one has a minmax regret model.

Between these two *extreme* optimization paradigms, expected/maximum regret costs, different *intermediate* optimization models can be generated from the framework of the problem (P) with the objective function (1). These new models could reflect, in a more suitable way, the specific characteristics of