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Utilizing null controllable regions to stabilize input-constrained nonlinear systems

Tyler Homer, Prashant Mhaskar*

Department of Chemical Engineering, McMaster University, Hamilton, Canada

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ABSTRACT

In this paper, we present a method for control of input-constrained nonlinear systems that offers guaranteed stabilization from the entire null controllable region (NCR). The controller achieves stabilization by using a constrained control Lyapunov function (CCLF) based on this NCR. Prior to online implementation, the level sets the CCLF are constructed using an iterative algorithm. The algorithm works by using an invariance principle to expand an initial quadratic Lyapunov function-based region of attraction. The level sets of this CCLF are then utilized in the control calculations, and in particular, an MPC is formulated that requires the system to go to lower level sets of the CCLF. The proposed MPC thus achieves stabilization from the entire NCR. The proposed approach is first corroborated against existing results for linear systems using two- and three-dimensional linear systems examples. Subsequently, the implementation is shown for two and three dimensional nonlinear systems.

1. Introduction

All real control systems possess physical constraints on the magnitude of their manipulated inputs. For example, control elements such as valves and variable-speed motors have limited open area and torque. Thus, a long-standing problem in nonlinear control theory has been to determine the set of initial conditions from which an input-constrained nonlinear system can be stabilized. Existing results on determining this so-called null controllable region (NCR) have focused on various special instances of the problem. In one set of results, unconstrained linear systems are considered. It has been shown that controllability to the origin is equivalent to showing that the controllability Grammian matrix has full rank (see e.g. Chen, 1999). Along the same lines, unconstrained control-affine nonlinear systems require the invertibility of a state-dependent matrix results in stabilization from the entire state space (Khalil, 2002).

However, the above techniques do not address controllability in the input-constrained setting. One of the early results in this direction is Teel (2002) where semi-stable systems, for which the NCR is unbounded, are considered. Conversely, in Hu et al. (2002), the null controllable region is determined for input-constrained unstable linear systems. In a recent set of results, controllers were designed that utilized the explicit NCR characterization in designing the control law, and possessed the NCR as the closed-loop region of attraction (Mahmood and Mhaskar, 2008, 2014). This was achieved via utilizing the NCR to construct the unique constrained control Lyapunov function (CCLF), and to in turn utilize it in the control design.

For nonlinear systems, while the determination of the NCR has remained an open problem, a powerful approach that has been used to estimate a controller's stability region is through control Lyapunov functions. Thus, control designs have been proposed for which the closed-loop stability region can be explicitly computed and such control designs range from purely algebraic control laws (Sontag, 1989) to those that employ optimization tools to compute the control law (Mhaskar et al., 2006; Mayne et al., 2000). However, it is not clear which Lyapunov function one should use to enable a large stability region. More specifically, there are no current methods to determine which particular choice of the Lyapunov function will give rise to a controller with a region of attraction equal to the NCR.

It is known that the problem of determining the boundary of the NCR is equivalent to the solution of the minimum-time control problem (Lewis, 2006); however, this formulation is not amenable to direct solution. Clearly, solving this problem at each individual state would be computationally intractable, whereas approaches that make use of the nonlinear Hamilton–Jacobi–Bellman PDE suffer from a lack of appropriate and tractable boundary conditions (Bardi and Falcone, 1990; Bobrow et al., 1985; Bressan, 2010). Fur-

* Corresponding author.
E-mail address: mhaskar@mcmaster.ca (P. Mhaskar).

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ther still, while the minimum-time function has been recognized as a Lyapunov function, it is often difficult to utilize its discontinuous solutions in a practical control design.

In yet another consideration, stabilization is sometimes desired in systems for which sustained implementation of large inputs is impossible or undesirable in practice. In other cases, when operating close to the boundary of the NCR, disturbances can pose a risk of catastrophic destabilization (Mahmood and Mhaskar, 2014). Hence, it is valuable to use a controller which deliberately steers away from the NCR.

Motivated by the above considerations, this paper presents the construction of a Lyapunov function, and subsequently a control design that incorporates this Lyapunov function, in which the closed-loop region of attraction is exactly equal to the null controllable region. The special and unique feature of this particular Lyapunov function is that anywhere the system is stabilizable, the control design is guaranteed to be compute the set of control moves to achieve stabilization. The manuscript is organized as follows: in Section 2 preliminary definitions are presented, whereas the main results form Section 3. The CCLF definition will be introduced first, then our algorithm for determining the NCR, followed by our control design. In Section 4, the algorithms are demonstrated using several simulation examples.

2. Preliminaries

We consider nonlinear process systems defined by the four-tuple $\Sigma = \{\Omega, \mathcal{T}, \mathcal{F}, \mathcal{U}\}$, which consists of (in the style of Lewis (2006)) the state-space $\Omega = \mathbb{R}^n$, the time interval $\mathcal{T} \in \mathbb{R}$ (here $\mathcal{T} = [0, \infty)$), the dynamics $\mathcal{F} : \Omega \times \mathcal{T} \times \mathcal{U} \mapsto \Omega$ given by:

$$\dot{x} = F(x, u) \tag{1}$$

and the control set $\mathcal{U} = [-1, 1] \subset \mathbb{R}$ so that $u(t) : \mathcal{T} \mapsto \mathcal{U}$ (Lewis, 2006). We assume that $x = 0$ is an isolated equilibrium point of the unforced system, i.e. $F(0, 0) = 0$ and that $F(\cdot)$ is globally Lipschitz continuous in all its arguments.

Let S and Q be sets. Then by S°, ∂S, $S \setminus Q$, we mean the interior, boundary and relative complement of S, respectively. Further, let $x(t; t_0, x_0, u)$ be the result of integrating a trajectory over time $t \in [t_0, t]$ under control law $u(t)$ from $x_0 = x(t_0)$.

By the null controllable region $\mathcal{C}$ (NCR) of system Σ, we mean the following set (Hu et al., 2002):

$$\mathcal{C} = \{x_0 : \exists u(t) \in \mathcal{U},\ T > t_0 \ \ s.t. \ \ x(T; t_0, x_0, u) = 0\} \tag{2}$$

Informally, $\mathcal{C}$ contains all the states that can be controlled to the origin with admissible controls. Importantly, the set $\mathcal{C}$ is the largest positively invariant set containing trajectories of (1) in reverse-time with the origin as the initial state. Thus, if $x(t_0 + t) \in \mathcal{C}$ and $t > 0$, then so is $x(t_0)$. Conversely, if $x(t_0) \notin \mathcal{C}$, then neither is $x(t_0 + t)$ for all $t > 0$. We will exploit this simple property in our NCR construction algorithm.

A function $V(x) : \mathbb{R}^n \mapsto \mathbb{R}$ is a control Lyapunov function if (i) $V(0) = 0$, and elsewhere, $V(x) > 0$, and (ii) there exists an admissible input function $u = \phi(x)$ such that $\dot{V}(x) = \frac{\partial V}{\partial x} F(x, \phi(x))$ is strictly negative on some ball surrounding the origin. If (ii) holds for all $x \in \mathcal{C}$, then we say the Lyapunov function $V(x)$ is a *constrained control Lyapunov function* (CCLF) (Mahmood and Mhaskar, 2014).

3. Results

3.1. Definition of the CCLF

There is more than one way to determine a CCLF. One example of a CCLF determination is to consider the time-optimal control problem. The set of states for which the problem with a particular value of the input yields a finite objective function yields a level set of the Lyapunov function. Below we utilize an alternate construction procedure for the determination of the level sets of this CCLF.

Consider the NCR for system Σ and denote $\Sigma(\mathcal{U}_k) = \{\Omega, \mathcal{T}, \mathcal{F}, \mathcal{U}_k\}$. Then, naturally $\mathcal{C}(\mathcal{U}_k)$ is the NCR (as in (2)) associated with $\Sigma(\mathcal{U}_k)$. Clearly, if $\mathcal{U}_2 \subset \mathcal{U}_1$, then $\mathcal{C}_2 \subseteq \mathcal{C}_1$, but this is one of the few comparison properties of $\mathcal{C}(\mathcal{U})$ that hold in the nonlinear setting (c.f. Mahmood and Mhaskar, 2014, where linearity is exploited).

By modulating the size of the control set $\mathcal{U}$, we can use the resulting NCR $\mathcal{C}(\mathcal{U})$ to generate the CCLF. In particular, lets consider only the family of intervals as our control sets; namely, let $\mathcal{I}(\lambda) = [-\lambda, \lambda]$ and, further, let $\mu(\mathcal{C}(\mathcal{I}(\lambda))) = \lambda$.

Thus, we suggest this constrained control Lyapunov function:

$$\mathcal{V}(x) = \sup\{\lambda : x \in \mathcal{C}(\mathcal{I}(\lambda))\} \tag{3}$$

We can deduce the following simple fact about $\mathcal{V}(x)$:

Proposition 1. *Suppose $x_0 \in \mathcal{C}^\circ$. Then, for some there must be a control law $u(t) = \phi(x(t))$ and time T such that $\mathcal{V}(x(T)) - \mathcal{V}(x_0) < 0$. Furthermore, $\lim_{t\to\infty} x(t; x_0, \phi(x)) = 0$.*

Proof: The existence of $\phi(x)$ is implied by the construction of the NCR (2); that is, there is always an input sequence to drive the states $x \in \mathcal{C}$ to the origin, and hence also to smaller level sets of $\mathcal{V}(x)$.□

The interpretation of the level set $\mathcal{V}(x) = c$ is that they are boundaries of regions in which the maximum control effort required is c, so that also $\mathcal{V}(x) = \sup_{t>0} |\phi(x(t))|$. We remark that when $F(\cdot)$ is convex, $\mathcal{V}(x)$ is differentiable. However, if Σ is a general non-linear system, then levels $\{x : \mathcal{V}(x) = c\}$ and $\{x : \mathcal{V}(x) = c + \varepsilon\}$, ε a small positive number, cannot be made arbitrarily close for every choice of c. In this sense, the scalar field $\mathcal{V}(x)$ contains discontinuities.

3.2. An algorithm to determine $\mathcal{C}$

The above-stated CCLF can only be utilized in control design after $\mathcal{C}$ is explicitly known. Thus, we now describe a construction algorithm that uses the positive invariance property of $\mathcal{C}$. An important assumption of this algorithm is that the NCR has a boundary consisting of states with strictly finite magnitudes (see Remark 5).

Step 1: We first construct an initial region of attraction using an arbitrary control Lyapunov function such as $V_i(x) = \sum_1^n x_i^2$. As in Khalil (2002), we first establish the sets $\Pi = \{x : \dot{V}_i(x) + \rho V_i(x) < 0\}$ and $\mathcal{V}(c) = \{x : V_i(x) \le c\}$. Then we define its *region of attraction* as just:

$$\mho = \sup_c\{x \in \mathcal{V}(c) : \mathcal{V}(c) \subseteq \Pi\} \tag{4}$$

In step one, we finitely discretize the domain into a grid $R \subset \mathbb{R}^n$. Let $R_y \subset \mathbb{R}^a$ be a *grid* if it is a collection of points $y \in \mathbb{R}^a$ that all admit the re-parameterization $y = h_y \delta_y$, where $h_y > 0$ is a scalar constant, the *grid spacing*, and $\delta_y \in \mathbb{N}^a$ is a vector of natural numbers. We say δ is the *index coordinates* of y if it is the unique δ_y such that $y = h_y \delta_y + r_y$, where $0 \le r_y < h$. Further, we define a *cell* associated with index δ to be the space $h\delta \le x - x_{grid,0} \le h(\delta + 1)$, $x_{grid,0}$ a chosen reference point, whereas the *cell vertices* are the corresponding indices.

We populate the grid C_0, the initial approximation of the NCR, with the index locations $R \cap \mho$, and the grid B_0, the initial approximation of $\partial\mathcal{C}$ (the boundary of the NCR) with the index locations $R \cap \partial\mho$.

Remark 1. In practice, the discrete approximation of B_0 should include some cells in $\mho^\circ$ that are close to, but not on, $\partial\mho$, resulting in a 'donut' of cells so that there are no 'gaps' caused by the discretization of the state space into $x \in R$. There is no harm to overestimating B_0 this way.