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Discrete Optimization

Integer programming models for the multidimensional assignment problem with star costs ☆

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ABSTRACT

We consider a variant of the multidimensional assignment problem (MAP) with decomposable costs in which the resulting optimal assignment is described as a set of disjoint stars. This problem arises in the context of multi-sensor multi-target tracking problems, where a set of measurements, obtained from a collection of sensors, must be associated to a set of different targets. To solve this problem we study two different formulations. First, we introduce a continuous nonlinear program and its linearization, along with additional valid inequalities that improve the lower bounds. Second, we state the standard MAP formulation as a set partitioning problem, and solve it via branch and price. These approaches were put to test by solving instances ranging from tripartite to 20-partite graphs of 4 to 30 nodes per partition. Computational results show that our approaches are a viable option to solve this problem. A comparative study is presented.

1. Introduction

The multidimensional assignment problem (MAP), originally introduced by Pierskalla (1968), aims to minimize the overall cost of assignment when matching elements from $\mathcal{N} = \{N_1, \dots, N_n\}$ $(n > 2)$ disjoint sets of equal size m. It comes as a natural generalization of the two-dimensional Assignment Problem (AP), known to be polynomially solvable (Edmonds & Karp, 1972; Kuhn, 1955). Among all the different generalizations of the MAP, the one considered in this paper is the *axial* MAP (hereafter referred to as MAP). In an axial MAP, each element of every set must be assigned to exactly one of m disjoint n-tuples, and each n-tuple must contain exactly one element of each set. Contrary to the AP, the MAP is known to be NP-hard (Karp, 2010) for all values of $n > 2$.

The MAP is usually presented as the following integer (0–1) program

$$(MAP): \ \min \sum_{i_1 \in N_1} \sum_{i_2 = N_2}, \dots, \sum_{i_n \in N_n} c_{i_1 i_2, \dots, i_n} x_{i_1 i_2, \dots, i_n} \tag{1}$$

$$\text{s.t.} \sum_{i_2 \in N_2} \sum_{i_3 \in N_3}, \dots, \sum_{i_n \in N_n} x_{i_1 i_2, \dots, i_n} = 1, \quad i_1 \in N_1 \tag{2}$$

$$\sum_{i_1 \in N_1}, \dots, \sum_{i_{s-1} \in N_{s-1}} \sum_{i_{s+1} \in N_{s+1}}, \dots, \sum_{i_n \in N_n} x_{i_1 i_2, \dots, i_n} = 1,$$
$$i_s \in N_s, \quad s = 2, \dots, n-1 \tag{3}$$

$$\sum_{i_1 \in N_1} \sum_{i_2 \in N_2}, \dots, \sum_{i_{n-1} \in N_{n-1}} x_{i_1 i_2, \dots, i_n} = 1, \quad i_n \in N_n \tag{4}$$

$$x_{i_1 i_2, \dots, i_n} \in \{0, 1\}, \quad i_s \in N_s, \quad s = 1, \dots, n, \tag{5}$$

where for every n-tuple $(i_1, i_2, \dots, i_n) \in N_1 \times N_2 \times \dots \times N_n$, variable $x_{i_1 i_2, \dots, i_n}$ takes the value of one if elements of the given n-tuple belong to the same assignment, and zero otherwise. The total assignment cost (1) is computed as the cost of matching elements from different sets together. As an example, an assignment which selects elements $(i_1, i_2, \dots, i_n)$ to be grouped together would have a cost of $c_{i_1 i_2, \dots, i_n}$.

Depending on the definition of the assignment costs, there are several variations of the MAP that can be considered. These variations are mainly associated with cases where the assignment cost of each n-tuple can be decomposed as a function of all possible pairwise assignment costs between elements of different sets. That is, $c_{i_1 i_2, \dots, i_n} = f(c_{i_1 i_2}, \dots, c_{i_n i_{n-1}})$, where $f: N_1 \times N_2 \cup \dots \cup N_{n-1} \times N_n \to \mathbb{R}$ and $c_{i_s i_t}$ is the cost of assigning together elements $i_s \in N_s$ and $i_t \in N_t$, for $s \neq t$. In general, the main advantage of having decomposable cost functions is that there may be ways of tackling the problem without having to completely enumerate all of the different assignment costs, which can be exponentially many. Moreover, most of these MAP variations can be associated with a weighted n-partite graph, in which the elements are represented by the vertices of the graph, each of the edges describes the decision of assigning two elements within the same n-tuple, and the

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weights on the edges account for the corresponding assignment costs. We provide a detailed explanation of this representation in Section 2.

Based on the applications and the context of the problem, there are different definitions of the MAP with decomposable costs that can be found in the literature (Aneja & Punnen, 1999; Bandelt, Crama, & Spieksma, 1994; Burkard, Rudolf, & Woeginger, 1996; Crama & Spieksma, 1992; Malhotra, Bhatia, & Puri, 1985; Kuroki & Matsui, 2009). In this paper we consider the case where each n-tuple of any feasible assignment is assumed to form a star (see, Bandelt et al., 1994). Nonetheless, since most of the extant literature concentrates on the case where the n-tuples form cliques, we also provide a brief description of the latter to emphasize the differences and enrich the discussion.

For the case of the cliques, a feasible assignment includes all possible pairwise connections within the elements of each tuple and thus, the cost of tuple $(i_1, i_2, \ldots, i_n) \in N_1 \times N_2 \times \cdots \times N_n$ is defined as the sum of all pairwise assignment costs. That is,

$$c_{i_1 i_2,\ldots,i_n} = \sum_{s=1}^{n} \sum_{t=s+1}^{n} c_{i_s i_t} \tag{6}$$

On the other hand, for the case of the stars, one element of each tuple is assigned to be a center (or representative) and the other elements are considered to be the leafs (or legs) of the star. Note that, contrary to the case of the cliques, each tuple can generate many different star configurations, depending on which element is selected as the center. Assuming for example, that the center is element i_s, the cost of the induced star is the sum of the pairwise costs between i_s and the other elements of the tuple. In view of these multiple possible configurations, the cost of tuple $(i_1, i_2, \ldots, i_n) \in N_1 \times N_2 \times \cdots \times N_n$ is defined as the minimum cost among the costs of all the possible star configurations of the tuple. That is,

$$c_{i_1 i_2,\ldots,i_n} = \min_{i_s \in \{i_1, i_2, \ldots, i_n\}} \left\{ \sum_{t \in \{1,2,\ldots,n\} \setminus \{s\}} c_{i_s i_t} \right\} \tag{7}$$

We name the aforementioned MAP version, the *multidimensional star assignment problem* (MSAP), because of the particular structure that each feasible assignment has. Despite the fact that this variant is often referred to as a particular case of the clique version (Bandelt et al., 1994), we consider that it is relevant to state it in a separate form. We base our argument on the fact that there exist applications for which the use of this variant could be of benefit. Moreover, there are formulations and techniques specifically tailored to solve the MSAP.

Several methodologies have been proposed to solve different variants and generalizations of the MAP, including exact approaches, approximation algorithms, heuristics, and metaheuristics. In particular, given the inherent NP-hardness of the problem, heuristic approaches have gained practical interest over the years. These include greedy heuristics (Balas & Saltzman, 1991), generalized randomized adaptive search procedures (GRASP) (Murphey, Pardalos, & Pitsoulis, 1999; Robertson III, 2001), GRASP with path relinking (Aiex, Resende, Pardalos, & Toraldo, 2005), randomized algorithms (Oliveira & Pardalos, 2004), genetic algorithms (Gaofeng & Lim, 2003), memetic algorithms (Karapetyan & Gutin, 2011b), local search heuristics (Bandelt, Maas, & Spieksma, 2004; Karapetyan & Gutin, 2011a), simulated annealing (Clemons, Grundel, & Jeffcoat, 2004), decomposition schemes (Vogiatzis, Pasiliao, & Pardalos, 2013), Lagrangian based procedures (Balas & Saltzman, 1991; Frieze & Yadegar, 1981; Poore & Robertson, 1997), and branch-and-bound techniques (Larsen, 2012; Pasiliao, Pardalos, & Pitsoulis, 2005).

From the perspective of approximation algorithms, there exist the works of Crama and Spieksma (1992) and Bandelt et al. (1994). Furthermore, contributions to the study of the polyhedral structure of the MAP formulation and other generalizations can be found in Appa, Magos, and Mourtos (2006), Balas and Saltzman (1989) and Magos and Mourtos (2009). Finally, studies related to the asymptotic behavior of the expected optimal value of the MAP, as well as tools to perform probabilistic analysis of MAP instances are given in Krokhmal, Grundel, and Pardalos (2007), Grundel, Oliveira, and Pardalos (2004) and Gutin and Karapetyan (2009).

Among all the proposed techniques listed above, we next focus our attention on approaches that are either proposed to tackle the MSAP, or that are designed to solve generalizations of the MAP, and thus can also be used to solve this problem. To solve the MSAP, Crama and Spieksma (1992) introduced an approximation algorithm designed to solve the three-dimensional case (i.e., $n = 3$). The proposed algorithm consists of sequentially solving two linear assignment problems. First, the elements of set N_1 are assigned to the ones of set N_2 and then, the resulting pairs are assigned to the elements of set N_3. The authors proved that if the pairwise assignment costs satisfy the triangle inequality, the proposed algorithm produces a $\frac{1}{2}$ approximation. Moreover, noting that this algorithm can produce three different solutions by simply varying the assignment order of the sets (e.g., assigning first the elements of N_1 to the ones of set N_3 and then, assigning the resulting pairs to the elements of set N_2), Crama and Spieksma proved that selecting the best of the three solutions yields a $\frac{1}{3}$ approximation.

In a subsequent study, Bandelt et al. (1994) proposed two type of heuristics, namely the hub and the recursive heuristics. They can be viewed as generalizations of the approach proposed by Crama and Spieksma (1992), but designed to solve the general n-dimensional case. The authors also provide an upper bound on the ratio between the cost of the solutions produced by these heuristics and the cost of the optimal solution.

As mentioned before, the MSAP is a particular case of the MAP and therefore, it can be solved using formulation (1)–(5). The polyhedral studies introduced by Balas and Saltzman (1989), for the three-dimensional case, and by Appa et al. (2006) and Magos and Mourtos (2009), for a more general version of the MAP, can be used to enhance (1)–(5) via the introduction of cutting planes. Using formulation (1)–(5) to solve the MSAP has one main drawback. It requires that all possible star costs be generated beforehand. This could be problematic because the total number of possible stars grows exponentially with the size of the problem (see Section 3). To circumvent this issue, it is possible to embed (1)–(5) within a branch-and-price scheme (see Section 3). Therefore, instead of enumerating all possible stars from the beginning, those are generated via column generation, in case they are considered suitable. The downside of this approach, though, is that mixing cutting planes and column generation is in general a difficult task (Barnhart, Johnson, Nemhauser, Savelsbergh, & Vance, 1998; Desaulniers, Desrosiers, & Spoorendonk, 2011; Lübbecke & Desrosiers, 2005).

For additional information about the MAP and its variations, we refer the reader to the surveys provided by Burkard, Çela, Pardalos, and Pitsoulis (1998), Burkard and Çela (1999), Burkard (2002), Gilbert and Hofstra (1988), Pardalos and Pitsoulis (2000), Pentico (2007) and Spieksma (2000).

This paper is inspired by the context of multi-sensor multi-target tracking problems, that involve the assignment of a series of sensor observations into a set of different targets. The relationship between these problems and the MAP, has been stated and studied by many authors including Bandelt et al. (2004), Chummun, Kirubarajan, Pattipati, and Bar-Shlom (2001), Deb, Pattipati, and Bar-Shalom (1993), Deb, Yeddanapudi, Pattipati, and Bar-Shalom