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Continuous Optimization

Constraint qualifications in convex vector semi-infinite optimization[☆]

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ABSTRACT

Convex vector (or multi-objective) semi-infinite optimization deals with the simultaneous minimization of finitely many convex scalar functions subject to infinitely many convex constraints. This paper provides characterizations of the weakly efficient, efficient and properly efficient points in terms of cones involving the data and Karush–Kuhn–Tucker conditions. The latter characterizations rely on different local and global constraint qualifications. The results in this paper generalize those obtained by the same authors on linear vector semi-infinite optimization problems.

1. Introduction

We consider convex optimization problems of the form

$$P : \text{“}\min\text{”}\ f(x) = (f_1(x), \ldots, f_p(x)) \text{ s.t. } g_t(x) \le 0, t \in T, \qquad (1)$$

where $x \in \mathbb{R}^n$ (the *space of decisions*), $f(x) \in \mathbb{R}^p$ (the *objective space*), the index set T is a compact Hausdorff topological space, $f_i : \mathbb{R}^n \to \mathbb{R}$ is a convex function, $i = 1, \ldots, p$, g_t is convex for each $t \in T$, and the function $(t, x) \mapsto g_t(x)$ is continuous on $T \times \mathbb{R}^n$. The continuity of f is consequence of the assumptions on its components $f_1, \ldots, f_p$. The model (1) includes ordinary convex (scalar and vector) optimization problems just taking the discrete topology on the (finite) index set. Since the optimality theory for this class of problems has been thoroughly studied, we assume in the sequel that T is infinite. When $p \geq 2$, P is a convex vector semi-infinite optimization (SIO in brief) problem; otherwise, P is a convex scalar SIO problem. Replacing in (1) the space of decisions $\mathbb{R}^n$ by an infinite dimensional space (typically a locally convex Hausdorff topological vector space) one gets a convex (scalar or vector) infinite optimization (IO in short) problem.

We assume throughout the paper that $p \geq 2$ and the *feasible set* of P, denoted by X, is non-empty. Obviously, X is a closed convex set whereas its image by the vector-valued objective function $f(X) \subset \mathbb{R}^p$ is possibly non-convex and non-closed. The vector SIO problem P can be reformulated as a vector optimization problem with the single convex constraint function $\varphi(x) := \max_{t \in T} g_t(x)$, called *marginal function:*

$$P : \text{“}\min\text{”}\ f(x) = (f_1(x), \ldots, f_p(x)) \text{ s.t. } \varphi(x) \le 0.$$

Throughout the paper we use the following notation. Given $x, y \in \mathbb{R}^m$, we write $x \leqq y$ ($x < y$) when $x_i \le y_i$ ($x_i < y_i$, respectively) for all $i = 1, \ldots, m$. Moreover, we write $x \le y$ when $x \leqq y$ and $x \neq y$.

An element $\overline{x} \in X$ is said to be *efficient* (*weakly efficient*) if there is no $\widehat{x} \in X$ such that $f(\widehat{x}) \le f(\overline{x})$ ($f(\widehat{x}) < f(\overline{x})$, respectively). There are many notions of proper efficiency in the literature, as those introduced by Geoffrion, Benson, Borwein and Henig. Since P is convex, all these concepts are equivalent to the proper efficiency in terms of linear scalarization (see, e.g., Ehrgott, 2005), so that we recall only Geoffrion's definition: a feasible point $\overline{x} \in X$ is said to be *properly efficient*if there exists $\rho > 0$ such that, for all $i = 1, \ldots, p$ and $\widehat{x} \in X$ satisfying $f_i(\widehat{x}) < f_i(\overline{x})$, there exists $j \in \{1, \ldots, p\}$ such that $f_j(\widehat{x}) > f_j(\overline{x})$ and $\dfrac{f_i(\overline{x}) - f_i(\widehat{x})}{f_j(\widehat{x}) - f_j(\overline{x})} \le \rho$.

We denote by X_{pE}, X_E, and X_{wE} the sets of properly efficient points, efficient points, and weakly efficient points of P, respectively. Obviously, $X_{pE} \subset X_E \subset X_{wE}$, with $X = X_{wE}$ whenever one component of f is identically zero, and $X = X_{pE}$ in the trivial case that f is the null function. Moreover, it is known that $f(X_{pE})$ is dense in $f(X_E)$ (Hartley, 1978; see also Ehrgott, 2005, Theorem 3.17).

Given a (possibly non-convex) vector SIO problem

$$P : \text{“}\min\text{”}\ f(x) \text{ s.t. } x \in X,$$

$\overline{x} \in X$ is said to be *locally* (*properly, weakly*) *efficient solution* of P if there exists a neighborhood $\mathcal{N}$ of $\overline{x}$ such that $\overline{x}$ is (properly, weakly) efficient solution of

$$P_{\mathcal{N}} : \text{“}\min\text{”}\ f(x) \text{ s.t. } x \in X \cap \mathcal{N}.$$

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Global and local concepts coincide in convex vector SIO thanks to the convexity of X and the componentwise convexity of f. For instance, if $\bar{x} \in X$ is not weakly efficient there exists $\widehat{x} \in X$ such that $f(\widehat{x}) < f(\bar{x})$; since $f(\lambda\widehat{x} + (1-\lambda)\bar{x}) < f(\bar{x})$ for all $\lambda \in]0, 1[$, with $\lambda\widehat{x} + (1-\lambda)\bar{x} \in X \cap \mathcal{N}$ for λ sufficiently small, $\bar{x}$ cannot be a locally weakly efficient solution of P. The argument is similar for efficient solutions while the equivalence can easily be proved for proper efficient solutions via scalarization. For this reason, in convex vector SIO, we can characterize the (proper, weak) efficiency on the basis of local information. The known tests for non-linear vector optimization classify a given $x \in X$ as locally (properly, weakly) efficient solution or not through conditions involving subsets of the objective space $\mathbb{R}^p$ or suitable scalarizations of P (see, e.g., Boţ, Grad, & Wanka, 2009; Ehrgott, 2005).

In this paper, on convex vector SIO, we give conditions for $\bar{x} \in X_{pE}$, $\bar{x} \in X_E$, and $\bar{x} \in X_{wE}$ which are expressed in terms of convex cones contained in the decision space $\mathbb{R}^n$ or in terms of the existence of Karush–Kuhn–Tucker (KKT in short) multipliers which can be computed from $\bar{x}$ and the data describing P.

As a general rule, to obtain a checkable necessary optimality condition for a given constrained optimization problem, one needs to assume some property of the constraint system called *constraint qualification* (CQ in short). We consider in this paper four CQs which extend those used in our previous paper (Goberna, Guerra-Vazquez, & Todorov, 2013) on constraint qualifications in linear vector SIO. The strongest one is the natural extension of the CQ introduced by M. Slater in a seminal work on scalar optimization published in 1950, which was adapted to linear scalar SIO by Charnes, Cooper and Kortanek in the 1960s. A weaker CQ for convex scalar SIO has been proposed in Li, Zhao, and Hu (2013). The locally Farkas–Minkowski CQ was first defined in Puente and de Serio (1999) for linear scalar SIO, and then extended to convex scalar SIO in Goberna and López (1998) and to convex scalar IO in Dinh, Goberna, López, and Son (2007). CQs weaker than the locally Farkas–Minkowski one have been introduced in Li et al. (2013), for convex SIO problems, and in Li, Ng, and Pong (2008), for convex IO problems. The local Slater CQ, introduced in Section 3 of this paper, seems to be new while the extended Kuhn–Tucker CQ was introduced in Tapia and Trosset (1994) for convex IO as an extension of that used by H.W. Kuhn and A.W. Tucker in Kuhn, Tucker, and Newman (1951) for ordinary non-linear optimization problems. Section 1 of Li et al. (2008) reviews the state of the art on CQs in convex scalar optimization. Some of the previous works also deal with the so-called regularity (or closedness qualification) conditions involving the objective function and the constraints (see, e.g., the recent papers Sun, Li, and Zhao (2013) and Sun (2014), dealing with IO problems with DC objective function and convex constraints, and references therein).

The stability of linear and non-linear scalar SIO has been investigated since the last 1980s from different perspectives, e.g., the pseudo-Lipschitz property and the lower and upper semicontinuity of the efficient set mapping under different types of perturbations, well-posedness, and generic stability (see, e.g., Chuong, Huy, & Yao, 2009, 2010a, 2010b; Fan, Cheng, & Wang, 2012; Todorov, 1996; Todorov & Tzeng, 1994), while the existing literature on optimality conditions for vector SIO and vector IO problems is surprisingly limited.

The main antecedent of this paper is Goberna et al. (2013), on linear vector SIO, which provides characterizations of the weakly efficient, efficient and properly efficient solutions in terms of cones involving the data and KKT conditions. In Caristi, Ferrara, and Stefanescu (2010), on a class of vector SIO problems involving differentiable functions whose constraints satisfy certain invex-type conditions and are required to depend continuously on an index t ranging on some compact topological space T, KKT conditions for $\bar{x} \in X_{pE}$, $\bar{x} \in X_E$ and $\bar{x} \in X_{wE}$ are given. In Guerra-Vazquez, Rückmann, Xu, Teo, and Zhang (2014), on non-convex differentiable vector SIO, the authors discuss constraint qualifications as well as necessary and sufficient conditions for locally weakly efficient points and present optimality conditions for properly efficient points in the senses of Geoffrion and of Kuhn et al. (1951). Finally, in Chuong and Kim (2014), on non-smooth vector IO problems posed on Asplund spaces whose index set T has no topological structure, necessary conditions as well as sufficient conditions for weakly efficient solutions are obtained appealing to the machinery of non-smooth analysis and a certain CQ, for non-convex systems introduced in Chuong et al. (2009), which can be seen as an extension of the so-called basic CQs introduced in Li and Ng (2005), for scalar IO problems posed in Banach spaces.

The convex vector SIO problems considered in this paper arise in a natural way in robust linear vector optimization. Indeed, consider an uncertain linear vector optimization problem

$$(LP) \quad \text{“}\min\text{”}\ (c_1^\top x, \ldots, c_p^\top x) \text{ s.t. } a_t^\top x \geq b_t,\ t \in T,$$

where T is a finite set, $c_i \in \mathcal{U}_i \subset \mathbb{R}^n, i = 1, \ldots, p$, and $(a_t, b_t) \in \mathcal{V}_t \subset \mathbb{R}^{n+1}$, $t \in T$. The *uncertainty sets* $\mathcal{U}_i, i = 1, \ldots, p$ are arbitrary non-empty sets while $\mathcal{V}_t, t \in T$ are non-empty compact sets. The *robust minmax counterpart* of (LP) (term coined in Ehrgott & Idec (2014)) enforces feasibility for any possible scenario and assumes that the cost of any (robust) feasible decision will be the worst possible, i.e., the problem to be solved is

$$\text{“}\min\text{”} \left(\max_{c_1 \in \mathcal{U}_1} c_1^\top x, \ldots, \max_{c_p \in \mathcal{U}_p} c_p^\top x \right) \text{ s.t. } a_t^\top x \geq b_t, \forall (a_t, b_t) \in \mathcal{V}_t, t \in T. \tag{2}$$

Observe that (2) is as (1), just taking $f_i(x) = \max_{c_i \in \mathcal{U}_i} c_i^\top x$ (i.e., the support function of $\mathcal{U}_i$), $i = 1, \ldots, p$, and expressing the constraints either as $b - a^\top x \leq 0$ for all $(a, b) \in \bigcup_{t \in T} \mathcal{V}_t$ (a compact index set) or as $g_t(x) \leq 0$, with $g_t(x) = \max\{b - a^\top x : (a, b) \in \bigcup_{t \in T} \mathcal{V}_t\}$ for all $t \in T$ (a finite index set equipped with the discrete topology).

This paper is organized as follows. Section 2 recalls basic concepts of convex analysis to be used later, applying some of them to characterize the so-called subdifferential cone and its interior, and to describe the relationships between several types of “tangent” cones which are closely related with the negative polar of the active cone. Section 3 extends to convex vector SIO four out of six constraint qualifications introduced in Goberna et al. (2013) for linear vector SIO. The two exceptions, the Farkas–Minkowski and the local polyhedral constraint qualifications, have not been considered in this paper as they are too strong in the convex framework. For methodological reasons, we give simple direct proofs of the lemmas in Section 3 even though most of them could be also obtained via linearization. The auxiliary Section 4 establishes different characterizations of the sets X_{pE}, X_E, and X_{wE} in terms of the subdifferential cone; these characterizations do not involve constraint qualifications, i.e., they are independent of the given representation of the closed convex feasible set X. Finally, Section 5 combines the results in Sections 3 and 4 to get characterizations of X_{pE}, X_E, and X_{wE} in terms of KKT multipliers. Here the proofs are necessarily direct as the objective functions are not linear. These results are applied to the robust linear vector optimization problem (LP).

2. Preliminaries

We start this section by introducing the necessary notations and concepts. Given $Z \subset \mathbb{R}^n$, int$Z$, cl$Z$, and bd$Z$ denote the *interior*, the *closure*, and the *boundary* of Z, respectively. The scalar product of $x, y \in \mathbb{R}^n$ is denoted by $x^\top y$, the Euclidean norm of x by $\|x\|$, the corresponding open ball centered at x and radius $\varepsilon > 0$ by $B(x, \varepsilon)$, and the zero vector by 0_n. We also denote by conv Z the *convex hull* of Z, while cone $Z := \mathbb{R}_+ \text{conv}Z$ denotes the *convex conical hull* of $Z \cup \{0_n\}$. If Z is a convex cone, its *positive* (*negative*) *polar cone*